Prolegomena to a Structuralist Reconstruction of Quantum Mechanics*

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Abstract

A structuralist "reconstruction sketch" of an idealized theory is provided. This theory, **QM**, has some essential features of quantum mechanics. **QM** is a theory about abstract "result-observation events", formal characterizations of interactions among physical systems and their results. **QM** is a stochastic theory and in the stochastic apparatus some features of "real life" quantum mechanics are recognizable. The result-observation events themselves exhibit neither essentially quantum mechanical features nor essentially physical features. At the level of the basic theory element **QM** is more like a specialization of probability theory than a physical theory. It is only at the level of specialization of the basic theory element that essentially physical and quantum mechanical features may be introduced. The account provides a "reconstruction sketch" rather than a reconstruction largely in that no account is given of physically interesting specializations. It also falls short of a full reconstruction in that the mathematical apparatus is restricted to finite structures.

Keywords: structuralism - quantum - mechanics - probability

Resumen

Se proporciona un "esquema de reconstrucción" estructuralista de una teoría idealizada. Esta teoría, **QM**, tiene algunas características esenciales de la mecánica cuántica. **QM** es una teoría acerca de "eventos de observación de resultados" abstractos, caracterizaciones formales de las interacciones entre los sistemas físicos y sus resultados. **QM** es una teoría estocástica y en el aparato estocástico son reconocibles algunas características de la mecánica cuántica de la "vida real". Los eventos de observación de resultados mismos no exhiben características esencialmente mecánico-cuánticas ni características esencialmente físicas. En el nivel del elemento básico de la teoría, **QM** es más como una especialización de la teoría de la probabilidad que una teoría física. Es recién en el nivel de especialización del elemento básico de la teoría que se pueden introducir características esencialmente físicas y de mecánica cuántica. En el presente artículo se ofrece un "esquema de reconstrucción" más que una reconstrucción, debido en gran medida a que no proporciona un tratamiento de las especializaciones físicas interesantes. Tampoco constituye una reconstrucción completa en tanto el aparato matemático se limita a estructuras finitas.

Palabras clave: estructuralismo - cuántica - mecánica - probabilidad

* Received: 10 January 2011. Accepted in revised version: 18 March 2011

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‡ The author is grateful for the comments of two anonymous reviewers on an earlier draft.

Metatheoria 1(2)(2011): 93-130. ISSN 1853-2322.

1. Introduction

The aim here is to provide a structuralist "reconstruction sketch" of a somewhat abstract, idealized theory which has some of the essential features of quantum mechanics. This is an abbreviated version of a longer work in which most of the mathematical detail and illustrative examples have been omitted.

The discussion presupposes acquaintance with the structuralist approach to reconstruction of empirical theories as described in detail in Balzer, Moulines & Sneed (1987), Sneed (1971), and in somewhat less detail in Stegmüller (1979). A summary discussion of this approach in a somewhat wider context is provided in Schmidt (2003). Within this framework, the focus is on sketching the *formal core* of a *basic theory element*

$$\langle \mathbf{M_{pp}}, \mathbf{M_p}, \mathbf{M}, \mathbf{C} \rangle \qquad [1\text{-}1]$$

where the components of this core are described respectively in [2], [3], [4] and [6]. Some specializations are considered informally in [5]. Intended applications will not be explicitly considered.

The theory is called "**QM**". **QM** is essentially a theory about abstract "result-observable events". Roughly, result-observable events are abstract, formal characterizations of interactions among physical systems and the results of these interactions. **QM** is a stochastic theory and it is mainly in the stochastic apparatus that some features of "real life" quantum mechanics can be recognized. The result-observable events themselves exhibit no essentially quantum mechanical features. Indeed, they exhibit no essentially physical features. At least at the level of the basic theory element [E-1-1] **QM** is more like a specialization of probability theory than a physical theory. Somewhat like the Cheshire Cat's grin, **QM** is quantum mechanics with everything but the probability theory and linear algebra removed.

It is only at the level of specialization of the basic theory element that essentially physical and quantum mechanical features may be introduced. The present account provides a "reconstruction sketch" rather than a reconstruction largely in the sense that no account is given of these physically interesting specializations. It also falls short of a full reconstruction in that the mathematical apparatus is restricted to finite structures.

The main ideas of **QM** are sketched here. This sketch is intended to indicate the basic features of **QM** to be elaborated below and suggest analogues to quantum mechanics. These analogues will be intuitively apparent only to those who have some knowledge of quantum mechanics. There is no attempt to provide this knowledge here. Within the framework of **QM** it is also possible to characterize linear operators assigned to observables, their eigenvalues and expectation values [4.2]. While this might make the resemblance between **QM** and some well known formulations of quantum mechanics more apparent, it does not appear to be the most efficient approach to a structuralist reconstruction. The basic idea of **QM** [1.1] and the formal constructs representing them [1.2] are sketched. An overview of the subsequent elaboration of these ideas is provided [1.3]. The unfamiliar notation in this section is explained in [3.1].

1.1 Basic idea

QM is a theory about result-observation events described by conjunctions roughly of the form

$$\underset{\sim}{\hat{o}}\,\underset{\sim}{\hat{\upsilon}} \qquad \hat{o} \in \hat{\upsilon} \tag{1-2}$$

where $\hat{\upsilon}$ and $\hat{o}$ are respectively an observable and a result of that observable. Observables $\hat{\upsilon}$ are partitions of the set of states of a system; results $\hat{o}$ are members of partitions $\hat{\upsilon}$. The meaning of the under-script '~' is explained below [2.2.3.3].

These results are associated with vectors

$$\left|\hat{o}\right\rangle \tag{1-3}$$

in a Hilbert space $\mathbb{H}$, a finite dimensional, complex inner product space. Values of certain conditional probabilities, fundamental to the theory, are determined by the inner product in the following way

$$\mathbf{p}\left(\underset{\sim}{\hat{o}}\,\middle|\,\underset{\sim}{\hat{\upsilon}}\,\underset{\sim}{\underset{\bullet}{\hat{o}}}\,\underset{\sim}{\underset{\bullet}{\hat{\upsilon}}}\right) = \left|\left\langle \hat{o} \middle| \underset{\bullet}{\hat{o}} \right\rangle\right|^2 . \tag{1-4}$$

The vectors $\left|\hat{o}\right\rangle$ and the space $\mathbb{H}$ in which they live are theoretical concepts in **QM**. The $\left|\hat{o}\right\rangle$ play a role in **QM** analogous to that of state vectors in quantum mechanics. The results appearing in vectors on the right side of [E-1-4] are associated with their corresponding observables in the conditional probability on the left side. These conditional probabilities are the non-theoretical concepts of **QM**.

The vector $\left|\hat{o}\right\rangle$ may be expressed as a linear combination (superposition) of basis vectors for any basis in $\mathbb{H}$. But, the notation '$\underset{\sim}{\hat{o}}\,\underset{\sim}{\hat{\upsilon}}$' keeps in view the "natural" bases in which $\left|\hat{o}\right\rangle$ is itself the single non-null basis vector in the expression.

That '$\underset{\sim}{\hat{o}}\,\underset{\sim}{\hat{\upsilon}}$' is a conjunction is essential to viewing

$$\left|\left\langle \hat{o} \middle| \underset{\bullet}{\hat{o}} \right\rangle\right|^2 \tag{1-5}$$

as the value of a *conditional* probability. Roughly, '$\underset{\sim}{\hat{o}}\,\underset{\sim}{\hat{\upsilon}}$' is understood to mean

'$\hat{o}$ is the result of observation of observable $\hat{\upsilon}$.'

The conditional probability can be given a more concrete interpretation by including a temporal index so that

$$\mathbf{p}\left(\underset{\sim t}{\hat{o}} \,\middle|\, \underset{\sim t}{\hat{\upsilon}}\ \underset{\sim t-1}{\underset{\bullet}{\hat{o}}}\ \ \underset{\sim t-1}{\underset{\bullet}{\hat{\upsilon}}}\right) = \left|\left\langle {}^{t}\hat{o} \,\middle|\, \underset{\bullet\ t-1}{\hat{o}}\right\rangle\right|^{2}. \qquad [1\text{-}6]$$

The under-script '~' appears in the probability expression to indicate that the entities are "events", appropriate arguments for probability functions. Thus, '$\underset{\sim t}{\hat{o}}\ \underset{\sim t}{\hat{\upsilon}}$ ' is understood to mean

'$\underset{\sim t}{\hat{o}}$ is the result of observation of $\underset{\sim t}{\hat{\upsilon}}$ at time t.'

The conditional probability is then

'the probability that $\underset{\sim t}{\hat{o}}$ results, given that an observation of $\underset{\sim t}{\hat{\upsilon}}$ occurs at t, and that $\underset{\sim t-1}{\hat{o}}$ was the result of observation of $\underset{\sim t-1}{\hat{\upsilon}}$ occuring at time t – 1.'

The notation (explained more fully in 2.2.3.3) works so that '$\underset{\sim t}{\hat{\upsilon}}$ ' and '$\underset{\sim t-1}{\hat{\upsilon}}$ ' generally denote observations of different observables, the 't' and 't – 1' serving to distinguish the observables.

Though it is not always made explicit, quantum mechanics may be viewed as dealing with conditional probabilities roughly of the form [E-1-6] where the conditioning proposition describes a "state preparation procedure" *and* an "observation" of a system in the prepared state. In the terminology of **QM**, a result-observation event is a *state preparation* and a observation event alone is an *observation*. A **QM** *observable* is analogous to the physical interpretation of a quantum mechanical observable, a Hilbert space operator.

An explicit expression of this view is provided by:

> [...] to assert that the state vector is **Ψ** can be regarded as implying that the system has undergone a corresponding state preparation procedure, which could be described in more detail but all the relevant information is contained in the specification of **Ψ**. (Ballentine 1986, pp. 885-886)

Much the following discussion may be viewed as nothing more than working out the formal details required to view quantum mechanical probabilities in this way. More explicitly, that "which *could be* described in more detail" *is*. In the course of doing this the precise sense in which "all the relevant information is contained...in **Ψ**" will become clear.

Roughly, **QM** observables do two things:

1) determine the structure of the result lattice [2.2.2], the fundamental non-theoretical construct of **QM**;
2) describe the probabilities treated by **QM** [2.3].

However, they do not play an explicit role in *determining* these probabilities [4].

Explicitly describing the probabilities treated by **QM** is essential to a structuralist formulation of the theory. It is required for formal description of the non-theoretical structures. Intuitively, it is required to say what the theory is "about". Examples will be provided to demonstrate that this view of probability statements is a plausible, internally consistent rendering of probability statements in contexts analogous to those appearing in standard expositions of quantum mechanics.

But, it will not be argued systematically that this view is adequate to all uses of probability statements in quantum mechanics.

Aside from plausibility and internal consistency, the chief virtue in this view is that quantum mechanical probabilities are simply special cases of the probability concept now standard in mathematical treatments of probability theory. This view has been sketched in (Sneed 1971) and elaborated in (Herbut 1992). Here, it is elaborated in somewhat more detail within the framework of a structuralist reconstruction (Balzer, Moulines & Sneed 1987). Another virtue is ontological austerity. All constructs are based on "systems" viewed a sets of "states" manipulated with familiar mathematical apparatus. Possibly, the chief vice in this view is the cumbersome notation required to describe it precisely.

1.2 Formal Constructs

Result-observation events are abstract models for events which may be intuitively conceived as interactions of object and apparatus systems with results which are interpreted as values of certain observables of the object system.

This conception is elaborated by considering somewhat abstract *system-observable structures*

$$\langle \mathbf{S}, \mathbf{O}^{\mathbf{S}} \rangle \qquad [1\text{-}7]$$

where **S** is a finite set and $\mathbf{O}^{\mathbf{S}}$ is a set of partitions of **S**. Members of **S**

$$\sigma \in \mathbf{S} \qquad [1\text{-}8]$$

are *states* of *system* **S**. Members

$$\upsilon \in \mathbf{O}^{\mathbf{S}} \subseteq \mathbf{POT(S)} \qquad [1\text{-}9]$$

of $\mathbf{O}^{s}$ are **S**-*observables*. Intuitively, these are partitions that can be "physically realized" in the sense that there are physical means of determining which member of the partition contains the state of **S** at a specific time.

Partitions of **S** are viewed as a lattice $\mathbf{V}^{\mathbf{S}}$ with "finer than" partial ordering $\lhd^{s}$. The set of observables $\mathbf{O}^{\mathbf{S}}$ is required to be "upwardly closed" under $\lhd^{s}$ in that

$$\upsilon \in \mathbf{O}^{\mathbf{S}},\ \upsilon \lhd^{s} \underset{\bullet}{\upsilon} \Rightarrow \underset{\bullet}{\upsilon} \in \mathbf{O}^{s}. \qquad [1\text{-}10]$$

It is characterized by a sub-set of maximal partitions $\hat{\mathbf{O}}^{s}$,

$$\hat{\mathbf{O}}^{s} \subseteq \mathbf{O}^{\mathbf{S}} \qquad [1\text{-}11]$$

the $\lhd^{s}$-finest partitions. Intuitively, the members of maximal partitions

$$\hat{\upsilon} \in \hat{\mathbf{O}}^{s} \qquad [1\text{-}12]$$

provide them maximally specific information about the state of **S** that can be **O**^{S}-observed.

Sets of Results for observable $\upsilon \in \mathbf{O}^{S}$ will be denoted generically by 'o' so that '$o\upsilon$' denotes an ordered pair consisting of an observable of υ together with the result o.

The set

$$\left|\mathbf{B}^{so}_{\upsilon}\right| = \{o | o \subseteq \upsilon\} \qquad [1\text{-}13]$$

may be given the structure of a Boolean algebra $\mathbf{B}^{so}_{\upsilon}$ whose order relation is determined by the sub-set ordering of $\left|\mathbf{B}^{so}_{\upsilon}\right|$.

The set

$$\left|\mathbf{L}^{s}_{o}\right| = \bigcup_{\upsilon \in \mathbf{O}^{s}} \upsilon \qquad [1\text{-}14]$$

may be given the structure of a *result lattice* $\mathbf{L}^{s}_{o}$ with an order relation derived from the sub-set relations on $\left|\mathbf{L}^{s}_{o}\right|$. The atoms of $\mathbf{L}^{s}_{o}$ are

$$\left|\mathbf{A}^{s}_{o}\right| = \bigcup_{\hat{\upsilon} \in \hat{\mathbf{O}}^{s}} \hat{\upsilon}\,. \qquad [1\text{-}15]$$

The central theoretical construct of **QM** is [ ], an embedding of the result lattice $\mathbf{L}^{s}_{o}$ into the lattice of sub-spaces $\mathbf{L}^{\mathbb{H}}$ of the Hilbert space $\mathbb{H}$, so that $[o]$ is a sub-space of $\mathbb{H}$. For the [ ]-image of atoms, $\hat{o} \in \left|\mathbf{A}^{s}_{o}\right|$, traditional notation for normalized vectors $\left|\hat{o}\right\rangle$ viewed as representing one dimensional subspaces $[\hat{o}]$ is employed.

Probability functions in **QM** are familiar Kolmogorov probabilities defined on a finite Boolean algebra. In theoretical structures of **QM**, conditional probabilities in the static case are determined by

$$\mathbf{p}\left(\underset{\sim t}{o} \middle| \underset{\sim t}{\upsilon}\ \underset{\sim t\text{-}1}{o}\ \underset{\sim t\text{-}1}{\upsilon}\right) = \mathbf{Tr}\left|[o_{t}]\right|\mathrm{W}\left|[\hat{o}_{t}]\right| \qquad [1\text{-}16]$$

where $|\mathrm{W}|$ is a *statistical operator* on $\mathbb{H}$ such that

$$\mathbf{Tr}\left|[o_{t\text{-}1}]\right|\mathrm{W}\left|[\hat{o}_{t\text{-}1}]\right| = 1. \qquad [1\text{-}17]$$

The overscript $\left|\hat{\mathrm{O}}\right|$ denotes the adjoint of operator $\left|\mathrm{O}\right|$. The notation departs from the usual and is explaned in 3.1 below.

This reduces to [E-1-6] in the case of one-dimensional subspaces where $|[o]|$ is the projection operator corresponding to the subspace $[o]$. A more general formulation is provided in [4.1].

1.3 Overview

The discussion of **QM** begins with the non-theoretical structures $\mathbf{M_{pp}}$ [2] which consist of probability functions over a Boolean algebra of descriptions of sequences result-observation events. Ultimately, **QM** determines the values of certain conditional probabilities for these probability functions. Roughly, these are the probability that a specific result obtains, given the occurrence of an observation event and the results of a previous observation event. The result descriptions which appear in these conditional probabilities have the structure of an orthomodular, orthocomplemented lattice, a *result lattice* $\mathbf{L}^{s}_{o}$ [2.2.2].

The central mathematical construct for quantum mechanics is a separable, infinite dimensional, complex Hilbert space [3.1]. More abstractly, the mathematical structure is the lattice of closed sub-spaces of such a Hilbert space, a modular, orthocomplemented lattice. Here, we restrict our attention to the special case of finite dimensional Hilbert spaces $\mathbb{H}$ and their lattice of sub-spaces. Theoretical structures $\mathbf{M_p}$ of **QM** [3] are essentially an embedding **[.]** of the non-theoretical result lattice $\mathbf{L}^{s}_{o}$ into the lattice of sub-spaces in such a way that the lattice structure is preserved [3.2].

The fundamental theoretical law determining the class of models M for **QM** **[4]** requires that the conditional probability values in the non-theoretical structures are determined by the inner product on the Hilbert space (generally, together with a statistical operator on the Hilbert space) into whose lattice of sub-space the non-theoretical non-theoretical projector lattice is **[.]**-embedded. Different **[.]**-embeddings result in different determinations.

Constraints **C** for the basic theory element for **QM** will be informally considered in [6].

2. Non-theoretical Structures: $\mathbf{M_{pp}}$[QM$\mathfrak{S}$]

The set of non-theoretical structures for **QM** is $\mathbf{M_{pp}}$[**QM**$\mathfrak{S}$]. Its members are of the form

$$\langle \mathrm{S}, \mathbf{T}, \mathbf{p} \rangle \qquad [2\text{-}1]$$

where **S** is a "system", **T** is an interval of integers interpreted as "time" and **p** is a conditional probability function on sequences of "result-observation" events, described in more detail below [2.2.3].

2.1 System structures: $\mathfrak{S}$

$$\mathfrak{S} = \langle \mathbb{S}, \approx, \iota, \circ, \mathbb{E}, \mathrm{O} \rangle \qquad [2\text{-}2]$$

where $\mathbb{S}$ is a set of *systems*, $\mathbf{S} \in \mathbb{S}$. Systems are *finite* sets of states, though $\mathbb{S}$ may be infinite. Members and sub-sets of system **S**

$$\sigma \in \mathbf{S} \in \mathbf{POT}(\mathbf{S}) \qquad [2\text{-}3]$$

are respectively *micro states* and *macro states* of **S**. Both are *states* of **S**. Intuitively, systems have histories in which **S** is in exactly one of its states at any time in its history.

Members $\mathbb{S}$ of are disjoint. Components $\approx$ and $\circ$ are respectively an equivalence relation and a concatenation operation on $\mathbb{S}$ [2.1.1]. ι is an isomorphism function for equivalent systems. Intuitively, $\approx$-equivalent systems are systems of the same kind, e.g. electrons. They have the same cardinality and the isomorphism ι describes what counts as the same state in distinct, but equivalent systems. $\mathbb{E}$ is a set of elementary systems from which all members of $\mathbb{S}$ are constructed via $\circ$-concatenation. **O** assigns to systems certain partitions of their states which are "observable" [2.1.2]. Here we will focus mainly on $\mathbb{S}$ and **O**. The remainder of the elements in the structure $\mathfrak{S}$ are essential for formulating the constraints **C**, which we treat only summarily and informally [6]. A full discussion of system structures would require consideration of the construction of $\mathbb{S}$ from $\mathbb{E}$. This is not provided here.

2.1.1 Concatenation

$\circ$-concatenation is simply set-theoretic cross product, $\times$, renamed to indicate its role in this context. Thus,

$$\circ \in \mathbf{SET}(\mathbb{S}^2, \mathbb{S}) \qquad [2\text{-}4]$$

such that, for all $\mathbf{S}^{\mathrm{I}}, \mathbf{S}^{\mathrm{II}} \in \mathbb{S}$,

$$\mathbf{S}^{\mathrm{I}\circ\mathrm{II}} = \mathbf{S}^{\mathrm{I}} \circ \mathbf{S}^{\mathrm{II}} = \mathbf{S}^{\mathrm{I}} \times \mathbf{S}^{\mathrm{II}}. \qquad [2\text{-}5]$$

States of concatenations correspond to states in their concatenates in the manner defined by

$$\mathbf{O}_{\mathrm{I}}^{\mathrm{I}\circ\mathrm{II}} \in \mathbf{SET}\left(\mathbf{POT}\left(\mathbf{S}^{\mathrm{I}}\right), \mathbf{POT}\left(\mathbf{S}^{\mathrm{I}\circ\mathrm{II}}\right)\right) \qquad [2\text{-}6]$$

such that, for all $^{\mathrm{I}} \in \mathbf{POT}(\mathbf{S}^{\mathrm{I}})$,

$${}_{\mathrm{I}}^{\mathrm{I}\circ\mathrm{II}} := \mathbf{O}_{\mathrm{I}}^{\mathrm{I}\circ\mathrm{II}}\left({}^{\mathrm{I}}\right) = {}^{\mathrm{I}} \times \mathbf{S}^{\mathrm{II}}. \qquad [2\text{-}7]$$

Intuitively, members of $\mathbf{O}_{\mathrm{I}}^{\mathrm{I}\circ\mathrm{II}}\left({}^{\mathrm{I}}\right)$ are sets of pairs, the first member of which is a member of a member of $^{\mathrm{I}}$ and the second member is any member of $\mathbf{S}^{\mathrm{II}}$. Still more intuitively, all you know about members of $\mathbf{O}_{\mathrm{I}}^{\mathrm{I}\circ\mathrm{II}}(^{\mathrm{I}})$ is that the first member of pairs it is in $^{\mathrm{I}}$.

2.1.2 Observables

The element **O** appearing in [E-2-2] is an **S**-observable structure [2.1.2.2], assignment of certain $\lhd^{s}$-structured sets of **S**-partitions [2.1.2.1] to members of $\mathbb{S}$.

2.1.2.1 S-partitions

The *partition lattice* of system **S**

$$\mathbf{V}^{\mathrm{S}} = \langle\, |\mathbf{V}^{\mathrm{S}}|,\ \lhd^{s},\ \triangledown^{s},\ \triangle^{s},\ \{\mathrm{S}\},\ \underset{\mathrm{v}}{\mathrm{S}}\,\rangle. \qquad [2\text{-}8]$$

Partitions in concatenations correspond to partitions in their concatenates in the manner defined by

$$\mathbf{V}_{\mathrm{I}}^{\mathrm{I}\circ\mathrm{II}} \in \mathbf{SET}\left(\left|\mathbf{V}^{\mathrm{I}}\right|,\left|\mathbf{V}^{\mathrm{I}\circ\mathrm{II}}\right|\right) \qquad [2\text{-}9]$$

$$\upsilon_{\mathrm{I}}^{\mathrm{I}\circ\mathrm{II}} := \mathbf{V}_{\mathrm{I}}^{\mathrm{I}\circ\mathrm{II}} = \left\{o^{\mathrm{I}} \times \mathbf{S}^{\mathrm{II}} \middle| o^{\mathrm{I}} \in \upsilon^{\mathrm{I}}\right\} = \left\{o_{\mathrm{I}}^{\mathrm{I}\circ\mathrm{II}} \middle| o^{\mathrm{I}} \in \upsilon^{\mathrm{I}}\right\}. \qquad [2\text{-}10]$$

Thus, members of $\mathbf{V}_{\mathrm{I}}^{\mathrm{I}\circ\mathrm{II}}(\upsilon^{\mathrm{I}})$ are sets of pairs, the first member of which is a member of a member of υ^{I} and the second member is any member of $\mathbf{S}^{\mathrm{II}}$.

Note that

$$\upsilon_{\mathrm{I}}^{\mathrm{I}\circ\mathrm{II}} \triangle^{\mathrm{I}\circ\mathrm{II}} \upsilon_{\mathrm{II}}^{\mathrm{I}\circ\mathrm{II}} \in |\mathbf{V}^{\mathrm{I}\circ\mathrm{II}}|. \qquad [2\text{-}11]$$

2.1.2.2 S-observable structures

For $\mathbf{S} \in \mathbb{S}$, the set of **S**-*observable structures* is

$$\begin{aligned}
&\mathbb{O}^{\mathrm{S}} = \{\mathbf{O}^{\mathrm{S}} = \langle\, |\mathbf{O}^{\mathrm{S}}|,\ \lhd^{s}\,\rangle \mid \\
&1)\ |\mathbf{O}^{\mathrm{S}}| \subseteq |\mathbf{V}^{\mathrm{S}}|; \\
&2)\ \upsilon \in |\mathbf{O}^{\mathrm{S}}|,\ \underset{\bullet}{\upsilon} \in |\mathbf{V}^{\mathrm{S}}|,\ \upsilon \lhd^{s} \underset{\bullet}{\upsilon} \Rightarrow \underset{\bullet}{\upsilon} \in |\mathbf{O}^{\mathrm{S}}|; \qquad [2\text{-}12] \\
&3)\ \lhd^{os} = \lhd^{s}|_{|\mathrm{O}^{s}|}\ \}.
\end{aligned}$$

Members of $\mathbb{O}^{\mathrm{S}}$ are partial orderings (members of $|\mathbf{PO}|$) whose elements are sets of **S**-partitions $|\mathbf{O}^{\mathrm{S}}|$ [E-2-12-1] upwardly closed under the partition refinement relation $\lhd^{s}$ [E-2-12-2], and ordered by the restriction of $\lhd^{s}$ to $|\mathbf{O}^{\mathrm{S}}|$ [E-2-12-3].

The set of $\mathbb{S}$-*observable structures* is

$$\mathbb{O}^{\mathbb{S}} = \{\mathbb{O}^{\mathrm{S}} \mid \mathbf{S} \in \mathbb{S}\} \qquad [2\text{-}13]$$

an **S**-*observable assignment*, a function

$$\mathbf{O} \in \mathbf{SET}(\mathbb{S}, \mathbb{O}^{\mathbb{S}}) \qquad [2\text{-}14]$$

such that, for all $\mathbf{S} \in \mathbb{S}$,

$$\mathbf{O}^{\mathbf{S}} := \mathbf{O}(\mathbf{S}) \in \mathbb{O}^{\mathbf{s}}. \qquad [2\text{-}15]$$

The element **O** appearing in [E-2-2] is an **S**-observable assignment.

Intuitively, observables are partitions that can be "empirically realized" in the sense that there are empirical means of determining which member of the partition contains the state of **S** at specific times in the history of **S**. Thus, if we can determine which member of υ contains the state of S and $\upsilon \lhd^{s} \underset{\bullet}{\upsilon}$, then we can determine which member of $\underset{\bullet}{\upsilon}$ contains this state. Members of observable υ have no numerical values. They are related to "quantum mechanical observables" in a manner described in 4.2 below.

The set of *terminal observables* for S is $|\widehat{\mathbf{O}}^{s}|$, the set of lower bounds for $\mathbf{O}^{s}$. The set of *maximal observables* is

$$|\hat{\mathbf{O}}^{S}| = \{\hat{\upsilon} \in |\widehat{\mathbf{O}}^{s}| \mid \#(\hat{\upsilon}) = \max\{\#(\widehat{\upsilon}) \mid \widehat{\upsilon} \in |\widehat{\mathbf{O}}^{s}|\}\}. \qquad [2\text{-}16]$$

The *dimension* of $\mathbf{O}^{S}$ is

$$\mathbf{dim}\mathbf{O}^{S} = \#(\hat{\upsilon}) \;\; \hat{\upsilon} \in |\mathbf{O}^{S}|. \qquad [2\text{-}17]$$

$\mathbf{O}^{s}$ is *homogeneous* just when

$$|\hat{\mathbf{O}}^{S}| = |\widehat{\mathbf{O}}^{s}|. \qquad [2\text{-}18]$$

2.1.3 Notation

It is sometimes convenient to regard members of **S** as well as members of $|\mathbf{O}^{s}|$ and their members to be indexed. Indexed members of **S** are denoted by

$$^{\mathrm{i}}\sigma^{\mathrm{s}} \in \mathrm{S} \qquad [2\text{-}19]$$

'$\upsilon^{\mathbf{s}}_{\mathbf{i}}$' denotes the **i**-th observable of system **S** so that

$$^{i}o^{\mathrm{s}}_{\mathrm{i}} \in \upsilon^{\mathrm{s}}_{\mathrm{i}} \qquad [2\text{-}20]$$

is element *i* of partition **i** of system S. Note that '$^{\mathrm{i}}\sigma^{\mathrm{s}}$' denotes a *micro state,* a member of S, while '$^{i}o^{\mathrm{s}}_{\mathrm{i}}$' denotes a *macro state*, a sub-set of S. Below the right superscript 'S' will be omitted where context makes the relevant system evident.

For compound systems, this notation becomes

$$^{\mathrm{ij}}\sigma^{\mathrm{I}\circ\mathrm{II}} = \langle {}^{\mathrm{i}}\sigma^{\mathrm{I}}, {}^{\mathrm{j}}\sigma^{\mathrm{II}} \rangle \in \mathrm{S}^{\mathrm{I}\circ\mathrm{II}} \qquad [2\text{-}21]$$

$$^{ij}o^{\mathrm{I}\circ\mathrm{II}}_{\mathrm{ij}} = {}^{i}o^{\mathrm{I}}_{\mathrm{i}} \times {}^{j}o^{\mathrm{II}}_{\mathrm{j}} \in \upsilon^{\mathrm{I}\circ\mathrm{II}}_{\mathrm{i:I}} \,\Delta^{\mathrm{I}\circ\mathrm{II}}\, \upsilon^{\mathrm{I}\circ\mathrm{II}}_{\mathrm{j:II}}\,. \qquad [2\text{-}22]$$

2.1.4 Interpretation

Intuitively, $\mathbb{S}$ is provided by models for some "underlying" empirical theory distinct from **QM**. For example, it might be that

$$\mathbb{S} = \mathbf{M}[\mathrm{CPM}] \tag{2-23}$$

where the underlying theory is classical particle mechanics. In the structuralist formalism, this situation would be represented by an inter-theoretical link. Where the structures in $\mathbf{S} \in \mathbb{S}$ are tuples of numerical valued functions over some domain, the obvious choice for the states of **S** is configurations of function values consistent with the relations among the functions in the structure in **M**[**CPM**]. While it is possible that $\mathbb{S}$ might be provided by a non-mechanical theory, e. g. thermodynamics, or even a non-physical theory, say from the behavioral sciences, no plausible "real life" examples spring readily to mind. The result-observable ontology and specific features of **QM** probabilities [2.3], thus far, appear to be *de facto* confined to mechanical theories.

This conception of the relation between **QM** and some underlying theory is somewhat like the "instrumentalist" view of quantum mechanics sometimes attributed to Heisenberg (Jammer 1966, p. 323 ff.). Indeed, the structuralist formulation of **QM** may be regarded as a defense/explication of this view in that the view is expounded within the framework of a general, coherent view of how empirical theories work (Balzer, Moulines & Sneed 1987).

A result-observation event may be intuitively conceived as an "observation process", an interaction between an "object system" **S** and some "apparatus system" $\mathbf{S}'$ associated with the partition υ whose results are observed. This appears appropriate for cases in which the underlying theory is a physical theory. In some very abstract way, it may be generally appropriate. However, the formal apparatus of **QM** and the result-observable ontology are logically independent of this intuitive conception. Indeed, **QM** lacks the formal apparatus to express these intuitive ideas and standard treatments of quantum mechanics do not go beyond intuitive discussion. One may, however, at the cost of considerable notational complexity, emend **QM** to accommodate these ideas. Such an emendation appears essential to raising the question of the comparability of **QM**-probabilities with an underlying hidden variable determinism. This is not undertaken here.

2.2 Algebraic structures

The central algebraic construct for **QM** is the $\mathbf{O}^{\mathrm{S}}$-result lattice $\mathbf{L}^{\mathrm{s}}_{\mathrm{o}}$. It is convenient to view $\mathbf{L}^{\mathrm{s}}_{\mathrm{o}}$ as constructed from Boolean algebras $\mathbf{B}^{\mathrm{s}}_{\upsilon}$ generated by **S**-observables $\mathbf{O}^{\mathrm{S}}$. The domain for probability functions relevant to **QM** is the

Boolean algebra of result-observable sequences $\mathbf{B}^{so}_{T}$. These sequences are composed of **S**-result-observable pairs.

2.2.1 Result boolean algebras

Let

$$\mathbf{B}^{s} = \langle \mathrm{POT(S)}, \subseteq^{s}, \bigcup^{s}, \bigcap^{s}, \sim^{s}, \mathrm{S}, \boldsymbol{\Lambda} \rangle \qquad [2\text{-}24]$$

the Boolean algebra of sub-sets of **S** and

$$\mathbf{B}^{s}_{\upsilon} = \langle |\mathbf{B}^{s}_{\upsilon}|, \subseteq^{s}_{\upsilon}, \bigcup^{s}_{\upsilon}, \bigcap^{s}_{\upsilon}, \sim^{s}_{\upsilon}, \mathrm{S}, \boldsymbol{\Lambda} \rangle \qquad [2\text{-}25]$$

the sub-Boolean algebra of $\mathbf{B}^{s}$ generated by partition $\upsilon \in |\mathbf{V}^{s}|$.

Note that

$$\upsilon_{j} \lhd^{s} \upsilon_{j'} \Rightarrow \mathbf{B}^{s}_{\upsilon_{j'}} \sqsubseteq_{\mathrm{Boole}} \mathbf{B}^{s}_{\upsilon_{j'}}. \qquad [2\text{-}26]$$

2.2.2 Result lattice

The the $\mathbf{O}^{s}$-*result lattice* is

$$\mathbf{L}^{s}_{o} = \langle |\mathbf{L}^{s}_{o}|, \sqsubseteq^{s}_{o}, \sqcup^{s}_{o}, \sqcap^{s}_{o}, \neg^{s}_{o}, \mathrm{S}, \boldsymbol{\Lambda} \rangle \qquad [2\text{-}27]$$

$$|\mathbf{L}^{s}_{o}| = \bigcup_{\upsilon \in |O^{s}|} \upsilon \qquad \sqsubseteq^{s}_{o} = \subseteq^{s} \cap |\mathbf{L}^{s}_{o}|^{2}. \qquad [2\text{-}28]$$

The operations are such that, for all $\mathbf{x}, \mathbf{y} \in |\mathbf{L}^{s}_{o}|$:

$$\mathbf{x} \bigcup^{s} \mathbf{y} = \mathbf{z} \in |\mathbf{L}^{s}_{o}| \Rightarrow \mathbf{x} \sqcup^{s}_{o} \mathbf{y} = \mathbf{z}$$

$$\text{otherwise } \mathbf{x} \sqcup^{s}_{o} \mathbf{y} = \mathbf{S};$$

$$\mathbf{x} \bigcap^{s} \mathbf{y} = \mathbf{z} \in |\mathbf{L}^{s}_{o}| \Rightarrow \mathbf{x} \sqcap^{s}_{o} \mathbf{y} = \mathbf{z} \qquad [2\text{-}29]$$

$$\text{otherwise } \mathbf{x} \sqcap^{s}_{o} \mathbf{y} = \boldsymbol{\Lambda}^{s};$$

$$\mathbf{x} = \sim^{s}\mathbf{y} \in |\mathbf{L}^{s}_{o}| \Rightarrow \mathbf{x} = \neg^{s}_{o} \mathbf{y}.$$

It can be shown that $\mathbf{L}^{s}_{o}$ is an orthomodular, orthocomplemented lattice (Birkhoff 1967, p. 52 ff.) and thus, generally non-distributive. The lattice dimension of $\mathbf{L}^{s}_{o}$ is

$$\mathbf{dim}_{\mathrm{OLAT}} \mathbf{L}^{s}_{o} = \mathbf{dim}\mathbf{O}^{s} = \max\{\#(\hat{\upsilon}) \mid \hat{\upsilon} \in |\hat{\mathbf{O}}^{s}|\}. \qquad [2\text{-}30]$$

Intuitively, to form $\mathbf{L}^{s}_{o}$, the Boolean algebras $\mathbf{B}^{so}_{\hat{\upsilon}}$, $\hat{\upsilon} \in |\hat{\mathbf{O}}^{s}|$, are "pasted together" with identical elements in different $\mathbf{B}^{so}_{\hat{\upsilon}}$ "overlapping". Each $\mathbf{B}^{so}_{\hat{\upsilon}}$ is a sub-lattice of $\mathbf{L}^{s}_{o}$. Members of each $\mathbf{B}^{so}_{\hat{\upsilon}}$ mutually commute (lattice theoretically); members of different $\mathbf{B}^{so}_{\hat{\upsilon}}$'s do not.

2.2.3 Result-observable sequence boolean algebra

Result-observable sequences [2.2.3.2] are sequences of result-observable pairs [2.2.3.1]. A result-observable sequence Boolean algebra $\mathbf{B}_{\mathrm{T}}^{\mathrm{so}}$ [2.2.3.3] serves as the domain for probability functions in **QM**.

2.2.3.1 Result-observable pairs

The set of **SO**-*result observable pairs* is

$$\Xi_{\mathrm{o}}^{\mathrm{s}} = \left\{ \left\langle {}^{i}o_{\mathrm{i}}^{\mathrm{s}}, \upsilon_{\mathrm{i}}^{\mathrm{s}} \right\rangle \middle| \upsilon_{\mathrm{i}}^{\mathrm{s}} \in \left| \mathrm{O}^{\mathrm{s}} \right| \right\}. \quad [2\text{-}31]$$

Members of $\Xi_{\mathrm{o}}^{\mathrm{s}}$ are members of $|\mathbf{L}_{\mathrm{o}}^{\mathrm{s}}|$ paired with all observables of which they are members. Generally, a single $o \in |\mathbf{L}_{\mathrm{o}}^{\mathrm{s}}|$ will be paired with many different observables. Intuitively, we think of **S**-observables $\upsilon \in |\mathrm{O}^{\mathrm{s}}|$ as physical systems that interact with system **S** yielding different "results", $o \in \upsilon$ depending on the state of **S**. The same result may be obtained from different observables. The physical motivation for this abstraction is something like the Stern-Gerlach experiment (Fano & Fano 1959, Feynman 1965, Ch. 5).

2.2.3.2 Result-observable sequences

Essentially, **QM** is a theory about probabilities of certain result-observable sequences associated with **S**. For $\mathbf{T} \in \mathbf{INT}(\mathbb{I})$, we characterize these sequences as **SO**-*result-observable sequences*.

$$\mathbf{\Pi}_{\mathrm{T}}^{\mathrm{so}} = \mathbf{SET}\left(\mathbf{T}, \Xi_{\mathrm{o}}^{\mathrm{s}}\right). \quad [2\text{-}32]$$

The brackets and comma will be suppressed in the notation for members of $\Xi_{\mathrm{o}}^{\mathrm{s}}$ so that

$${}^{i}o_{\mathrm{i}}\upsilon_{\mathrm{i}} := \langle {}^{i}o_{\mathrm{i}}, \upsilon_{\mathrm{i}} \rangle \in \Xi_{\mathrm{o}}^{\mathrm{s}} \quad [2\text{-}33]$$

where the index 'i' of the observable υ and the index '*i*' of a specific result ${}^{i}o_{\mathrm{i}}$ of υ have been suppressed. Or, more concisely, when it is not necessary to specify the result

$$o\upsilon := \langle o, \upsilon \rangle, \; o \in \upsilon. \quad [2\text{-}34]$$

Note that the "null" observable {**S**} and its single result **S** appear as

$$\mathrm{S}\{\mathrm{S}\} \quad [2\text{-}35]$$

the null result-observable pair. Intuitively, this is understood to be associated with an event in which no observation of **S** occurs, i.e. **S** is "isolated" or "undisturbed".

The generic member of $\mathbf{\Pi}_{\mathrm{T}}^{\mathrm{so}}$ will be denoted by

$$\underline{o} \in \mathbf{\Pi}_{\mathrm{T}}^{\mathrm{so}} \quad [2\text{-}36]$$

so that, for all $\mathbf{t} \in \mathbf{T}$, there is some ${}^{i}o_{\mathrm{i}}\upsilon_{\mathrm{i}} \in \Xi_{\mathrm{o}}^{\mathrm{s}}$ such that

$$\underline{o}_{t} := \underline{o}(t) = {}^{i}o_{i:t}\upsilon_{j:t} = o_{t}\upsilon_{t}\,. \qquad [2\text{-}37]$$

2.2.3.3 Boolean algebra: $\mathbf{B}_{T}^{so}$

$\mathbf{B}_{T}^{so}$ is the *algebra of* **SO**-*result-observable sequences* where

$$|\mathbf{B}_{T}^{so}| = \mathrm{POT}(\mathbf{\Pi}_{T}^{so}). \qquad [2\text{-}38]$$

Of particular interest are the members of $|\mathbf{B}_{T}^{so}|$:

$$\begin{aligned} {}^{i}\underset{\sim\, i:t}{o} &:= \left\{\underline{o} \in \mathbf{\Pi}_{T}^{so} \,\middle|\, \pi_{1}(\underline{o}_{t}) = {}_{i}o_{i}\right\} \\ \underset{\sim\, i:t}{\upsilon} &:= \left\{\underline{o} \in \mathbf{\Pi}_{T}^{so} \,\middle|\, \pi_{2}(\underline{o}_{t}) = \upsilon_{i}\right\}. \end{aligned} \qquad [2\text{-}39]$$

For these, juxtaposition denotes set theoretic intersection (conjunction). For example,

$$\underset{\sim t}{o}\,\underset{\sim t}{\upsilon} := \underset{\sim t}{o} \cap_{T} \underset{\sim t}{\upsilon}\,. \qquad [2\text{-}40]$$

Note that

$$\underset{\sim t}{o}\,\underset{\sim t}{\upsilon} = \underset{\sim t}{\upsilon}\,\underset{\sim t}{o}\,. \qquad [2\text{-}41]$$

and that expressions like '$\underset{\sim t}{o}\,\underset{\sim t\text{-}1}{\upsilon}$' are also well defined as members of $|\mathbf{B}_{T}^{so}|$. Likewise, the expressions '$\underset{\sim i:t}{\hat{\upsilon}}\,\underset{\sim j:t}{\hat{\upsilon}}$' and '${}^{i}\underset{\sim\, i:t}{o}\,\underset{\sim j:t}{\upsilon}$' are well defined. However,

$$\begin{aligned} &\underset{\sim i:t}{\hat{\upsilon}}\,\underset{\sim j:t}{\hat{\upsilon}} = \mathbf{\Lambda} \\ &\mathbf{i} \neq \mathbf{j},\ {}^{i}o_{i} \not\ni \upsilon_{i} \cap \upsilon_{j} \Rightarrow {}^{i}\underset{\sim i:t}{o}\,\underset{\sim i:t}{\upsilon} = \mathbf{\Lambda} \end{aligned} \qquad [2\text{-}42]$$

Intuitively, '$\underset{\sim t}{o}\,\underset{\sim t}{\upsilon}$' denotes a "result-observation event", or an "observation at **t** of observable υ with result o". Generally, symbols with the under-script '~' denote "events", elements of the Boolean algebra $\mathbf{B}_{T}^{so}$. Symbols stripped of this under-script denote observables and their results associated with these events. Formal discussion required to explain rigorously this notational distinction is omitted.

In this notation,

$$\gamma_{[t\text{-}(\Delta t+1),t]} := \underset{\sim t}{o}\,\underset{\sim t}{\upsilon} \ldots \underset{\sim t\text{-}i}{o}\,\underset{\sim t\text{-}i}{\upsilon} \ldots \underset{\sim t\text{-}(\Delta t+1)}{o}\,\underset{\sim t\text{-}(\Delta t+1)}{\upsilon} \quad \Delta t \in \mathbb{I}^{[0,t\text{-}(\Delta t+1)} \in \mathbf{T}, \mathbf{i} \in \mathbb{I}^{[0,t\text{-}\Delta t)} \qquad [2\text{-}43]$$

is a *sequence* of result-observation events. The '**t-i**' subscript indicates the position of the result observation event $\underset{\sim t\text{-}i}{o}\,\underset{\sim t\text{-}i}{\upsilon}$ in the sequence. The same sequence would be denoted by any permutation of the order of the conjuncts e.g.

$$\underset{\sim t}{o}\,\underset{\sim t}{\upsilon}\,\underset{\sim t\text{-}1}{o}\,\underset{\sim t\text{-}1}{\upsilon} = \underset{\sim t\text{-}1}{\upsilon}\,\underset{\sim t}{o}\,\underset{\sim t}{\upsilon}\,\underset{\sim t\text{-}1}{o}\,. \qquad [2\text{-}44]$$

For intuitive clarity, the conjuncts appear in decreasing order in the sequence with result preceding observations.

2.3 Probabiltiy functions

Probabilities appearing in non-theoretical structures for **QM** [E-2-1] are

$$\mathbf{p} \in \mathbf{PROB}(\mathbf{B}^{so}_{T}) \qquad [2\text{-}45]$$

where **PROB(B)** is the set of all probability functions over the Boolean algebra **B**. Certain conditional probabilities derived from these are those determined by **QM**. These conditional probabilities need to be identified in a way that is completely explicit about their arguments. Doing this is effected using the rather cumbersome notation developed in [2.2.3.3].

Conditional probabilities determined by **QM** are those of the form

$$\mathbf{p}\left(\underset{\sim}{o}_{t} \middle| \underset{\sim}{\upsilon}_{t} \ldots \underset{\sim}{o}_{t\text{-}i}\, \underset{\sim}{\upsilon}_{t\text{-}i} \ldots \underset{\sim}{o}_{t\text{-}\Delta t}\, \underset{\sim}{\upsilon}_{t\text{-}\Delta t}\, \underset{\sim}{o}_{t\text{-}(\Delta t+1)}\, \underset{\sim}{\upsilon}_{t\text{-}(\Delta t+1)} \right). \qquad [2\text{-}46]$$

These are *projection probabilities*. It is important to understand that the *order* of the result observation events [E-2-43] is decisive in indicating which conditional probabilities are determined by **QM**, though the order of the conjuncts in the condition is not. Thus,

$$\mathbf{p}\left(\underset{\sim}{o}_{t} \middle| \underset{\sim}{\upsilon}_{t}\, \underset{\sim}{o}_{t\text{-}1}\, \underset{\sim}{\upsilon}_{t\text{-}1}\, \underset{\sim}{o}_{t\text{-}2}\, \underset{\sim}{\upsilon}_{t\text{-}2} \right) = \mathbf{p}\left(\underset{\sim}{o}_{t} \middle| \underset{\sim}{\upsilon}_{t}\, \underset{\sim}{\upsilon}_{t\text{-}1}\, \underset{\sim}{o}_{t\text{-}1}\, \underset{\sim}{o}_{t\text{-}2}\, \underset{\sim}{\upsilon}_{t\text{-}2} \right) \qquad [2\text{-}47]$$

is determined by **QM** while

$$\mathbf{p}\left(\underset{\sim}{o}_{t\text{-}1} \middle| \underset{\sim}{o}_{t}\, \underset{\sim}{\upsilon}_{t}\, \underset{\sim}{\upsilon}_{t\text{-}1}\, \underset{\sim}{o}_{t\text{-}2}\, \underset{\sim}{\upsilon}_{t\text{-}2} \right) \text{ and } \mathbf{p}\left(\underset{\sim}{o}_{t}\, \underset{\sim}{o}_{t\text{-}1} \middle| \underset{\sim}{\upsilon}_{t}\, \underset{\sim}{\upsilon}_{t\text{-}1}\, \underset{\sim}{o}_{t\text{-}2}\, \underset{\sim}{\upsilon}_{t\text{-}2} \right) \qquad [2\text{-}48]$$

though well defined as probability expressions (the arguments of the **p**-function being members of $|\mathbf{B}^{so}_{T}|$), *are not* determined by **QM**. Likewise, all the probability expressions in

$$\mathbf{p}\left(\underset{\sim}{o}_{t} \middle| \underset{\sim}{\upsilon}_{t}\, \underset{\sim}{o}_{t\text{-}1}\, \underset{\sim}{\upsilon}_{t\text{-}1} \right) := \mathbf{p}\left(\underset{\sim}{o}_{t}\, \underset{\sim}{\upsilon}_{t}\, \underset{\sim}{o}_{t\text{-}1}\, \underset{\sim}{\upsilon}_{t\text{-}1} \right) / \, \mathbf{p}\left(\underset{\sim}{\upsilon}_{t}\, \underset{\sim}{o}_{t\text{-}1}\, \underset{\sim}{\upsilon}_{t\text{-}1} \right) \qquad [2\text{-}49]$$

are well defined, but only the expression on the left side is determined by **QM**. **QM** determines these conditional probabilities without determining separately the components of the customary definition of conditional probabilities. Of course, **QM** does put limits on the values or the probabilities on the right. It will be seen below [4.1] that the specific way in which these conditional probabilities are determined by **QM** depends essentially on the order of the result observation events in [E-2-43].

Letting sequence right-to-left indicate temporal sequence, using the notation of [2.1.3] and being imprecise about the sequence of the right sub-scripts, [E-2-46] becomes

$$\mathbf{p}\left({}^{i}\underset{\sim}{o}_{i} \middle| \underset{\sim}{\upsilon}_{i} \ldots {}^{j}\underset{\sim}{o}_{j}\, \underset{\sim}{\upsilon}_{j} \ldots {}^{k}\underset{\sim}{o}_{k}\, \underset{\sim}{\upsilon}_{k} \right) \qquad [2\text{-}50]$$

Further, letting the right subscripts on the '$^{j}\underset{\sim j}{o}$''s indicate a temporally preceding '$\underset{\sim j}{v}$', it becomes

$$\mathbf{p}\left({}^{i}\underset{\sim i}{o}\,\middle|\,\ldots{}^{j}\underset{\sim j}{o}\ldots{}^{k}\underset{\sim k}{o}\right). \qquad [2\text{-}51]$$

It will become apparent below [4.1] that [E-2-51] is roughly the way probabilities are commonly viewed in discussions of quantum mechanics (Ballentine 1986, pp. 885-886, quoted above [1.1]). Quantum mechanical probabilities are indeed usually taken to be conditional probabilities, but the explicit nature of the conditioning event is suppressed in the notation. This considerably simplifies the notation and suffices for most purposes. But, certain questions of interpretation ([5.1.1.1.3], [4.2]) are clarified by more explicit notation.

QM is a theory *about* probabilities as described in modern treatments of mathematical probability theory. Intuitively, these are probabilities of individual events which are determined, in some way, by facts about these individual events. However, **QM** has nothing to say about these individual events nor the way in which facts about them determine probabilities. More specifically, **QM** has nothing to say about how one might come to know about result-observation sequences that appear in these conditional probabilities nor about how this knowledge is related to the probability function values. In this sense **QM** is "incomplete". Usual formulations of quantum mechanics are also "incomplete" in this way. This is viewed by some as a defect and efforts are made to remedy it (von Neumann 1955, Ch. VI, Sneed 1964, Chs. IV, V, Lombardi & Castagnino 2008). This matter is not considered further here.

More generally, **QM** is not committed to any specific view about the appropriate "interpretation" (e.g. personalistic, relative frequency, ensemble, propensity) of probability function values. There is a large body of general philosophical, as well as specifically quantum mechanical, literature pertaining to this question. No contribution to this literature is offered here. However, it may be useful to note that the personalistic interpretation, long defended by Jaynes (2003) as appropriate to statistical mechanical probabilities, has recently been suggested to be uniquely appropriate to quantum mechanics (Caves, Fuchs & Schack 2002).

3. Theoretical structures: $\mathbf{M_p}[\mathbf{QM}\mathfrak{S}]$

The set of theoretical structures for **QM** is $\mathbf{M_p}[\mathbf{QM}\mathfrak{S}]$. Its members are of the form

$$\langle \mathrm{S}, \mathbf{T}, \mathbf{p}, \mathbb{H}, [\,], \mathfrak{V} \rangle \qquad [3\text{-}1]$$

where non-theoretical structures

$$\langle \mathrm{S}, \mathbf{T}, \mathbf{p} \rangle \in \mathbf{M_{pp}}[\mathbf{QM}\mathfrak{S}] \qquad [3\text{-}2]$$

are emended with a Hilbert space $\mathbb{H}$, a lattice morphism **[ ]** which maps $|\mathbf{L}^{s}_{o}|$ many-one into $|\mathbf{L}^{\mathbb{H}}|$, the lattice of sub-spaces of $\mathbb{H}$, and $\mathfrak{V}$, a set of unitary operators on $\mathbb{H}$. Members of $\mathfrak{V}$ are dynamic operators, analogous to the Hamiltonian operator, which contribute to determining projection probabilities when null result observation events [E-3-35] occur. In the customary jargon of quantum mechanics, dynamic operators determine the change over time in the state vector of an isolated system.

Members of $\mathfrak{V}$ have no non-theoretical "correlates"; they are fully theoretical. This is a consequence of the austerity of the system concept in **QM**. When this concept is filled out to something like a physical theory, features of this theory will very likely constrain the choice of dynamic operators.

3.1 Hilbert structures: $\mathbb{H}$

The Hilbert space

$$\mathbb{H} = \langle \mathbb{V}_{\rangle}, \langle\rangle \rangle \qquad [3\text{-}3]$$

is a vector space $\mathbb{V}_{\rangle}$ over the field of complex numbers $\mathbb{C}$ supplied with and inner product $\langle\rangle$. We require that the dimension of $\mathbb{H}$ is that of $\mathbf{O}^{s}$, i.e

$$\mathbf{dim}_{\mathrm{VEC}}(\mathbb{V}_{\rangle}) = \mathbf{dim}\mathbf{O}^{s} \qquad [3\text{-}4]$$

[E-3-17] which is finite since **S** is a finite set.

For the most part, familiar notation for constructs related to $\mathbb{H}$ is employed. Exceptions are: '$|\mathbf{O}|$' denotes generic linear operators on $\mathbb{H}$; '$|\mathbf{P}|$' projection operators; '$|\mathbf{W}|$' statistical operators; '$|\hat{\mathbf{O}}|$' the adjoint of '$|\mathbf{O}|$'; '$|\mathbf{I}|$' is the identity operator. Adjacent '|''s are merged in operator application and multiplication; thus, we write '$|\mathbf{O}|\mathbf{v}\rangle$' and '$|\mathbf{O}|\mathbf{O}|$'. These exceptions are motivated by notational symmetry and harmony with the Dirac bra-ket notation. The use of overscript '^' in '$|\hat{\mathbf{O}}|$' is totally unrelated to the use of the same notational device in '$\hat{o}$'.

In contrast to most familiar expositions of quantum mechanics, rather than $\mathbb{H}$ itself, the theoretical structure if **QM** focuses on the lattice of sub-spaces of $\mathbb{H}$ (Halmos 1958, p. 16 ff.)

$$\mathbf{L}^{\mathbb{H}} = \langle |\mathbf{L}^{\mathbb{H}}|, \sqsubseteq^{\mathbb{H}}, \sqcup^{\mathbb{H}}, \sqcap^{\mathbb{H}}, \dashv^{\mathbb{H}}, \hat{0}, \mathbb{H} \rangle. \qquad [3\text{-}5]$$

It is a modular, orthocomplemented lattice (Birkhoff 1967, p. 52 ff.). $\mathbf{L}^{\mathbb{H}}$ may be visualized intuitively as an infinite number of "overlapping" Boolean algebras each corresponding to a basis for $\mathbb{H}$.

3.2 [ ]-Embeddings

A [ ]-*embedding* is a dimension preserving, orthocomplemented lattice morphism

$$[\ ] \in \mathbf{OLAT}(\mathbf{L}^{s}_{o}, \mathbf{L}^{\mathbb{H}}) \quad [3\text{-}6]$$

such that, for all $o \in |\mathbf{L}^{s}_{o}|$,

$$\mathbf{element\text{-}dim}_{\mathrm{OLAT}}(o) = \mathbf{sub\text{-}space}^{\mathbb{H}}\text{-}\mathbf{dim}([o]). \quad [3\text{-}7]$$

Some additional notation is convenient. For all $o \in |\mathbf{L}^{s}_{o}|$, let

$$[o] := [\ ](o) \qquad |[o]| := |\mathbf{P}|_{[o]} \quad [3\text{-}8]$$

where $|\mathbf{P}|_{[o]}$ is the projection operator for $\mathbb{H}$ in corresponding to the sub-space $[o]$. The [ ]-image of $\mathbf{L}^{s}_{o}$ in $|\mathbb{V}\rangle|$ is denoted by '$[\mathbf{L}^{s}_{o}]$' and the set of projection operators corresponding to members of $[\mathbf{L}^{s}_{o}]$ by '$|[\mathbf{L}^{s}_{o}]|$' so that

$$|[\mathbf{L}^{s}_{o}]| \subset \mathfrak{P} \quad [3\text{-}9]$$

the set of projection operators for $\mathbb{H}$.

For $\hat{\upsilon} \in \left|\hat{\mathrm{O}}^{s}\right|$,

$$[\hat{\upsilon}] := \{[\hat{o}] | \hat{o} \in \hat{\upsilon}\} \quad [3\text{-}10]$$

is a set of mutually orthogonal (as sub-spaces) one-dimensional subspaces of $\mathbb{H}$, atoms of $\mathbf{L}^{\mathbb{H}}$.

Generic *normalized* vectors in these sub-spaces are denoted by

$$|\hat{o}\rangle \in [\hat{o}]. \quad [3\text{-}11]$$

Thus, for $\hat{\upsilon}$, $\underset{\bullet}{\hat{\upsilon}} \in \left|\hat{\mathrm{O}}^{s}\right|$, inner products appear as

$$\langle \hat{o} | \hat{o} \rangle. \quad [3\text{-}12]$$

For $\hat{o}^{i}$, $\hat{o}^{j} \in \hat{\upsilon}$, $\hat{o}^{i} \neq \hat{o}^{j}$, $|\hat{o}^{i}\rangle$ and $|\hat{o}^{j}\rangle$ are orthogonal (as vectors), i.e.

$$\langle_{i} \hat{o} | \hat{o}^{j} \rangle = 0. \quad [3\text{-}13]$$

The normalization requirement is that

$$\langle_{i} \hat{o} | \hat{o}^{i} \rangle = 1. \quad [3\text{-}14]$$

Requiring that the $|\hat{o}\rangle$ to be normalized simplifies some notation by allowing the omission of a "normalization factor".

The Kochen-Specker theorem (Held 2008) may be viewed, in part, as describing limitations on configurations of sub-spaces in $|\mathbf{L}^{\mathbb{H}}|$ that can be the images of a [ ]-embedding. However, conclusions frequently drawn from these limitations require further assumptions which are not true for **QM** [4.2].

4. Theoretical laws: $\mathbf{M}[\mathbf{QM}\mathfrak{S}]$

For theoretical structures of the form

$$\langle \mathrm{S}, \mathbf{T}, \mathbf{p}, \mathbb{H}, [\ ], \mathfrak{V} \rangle \in \mathbf{M_p}[\mathbf{QM}\mathfrak{S}] \qquad [4\text{-}1]$$

the basic theoretical law of **QM** requires that the probability functions **p** [2.3] are partially determined by the inner product on the Hilbert space $\mathbb{H}$ via the embedding function **[ ]**. The projection probability values [E-2-46] for maximal observables and $\Delta\mathbf{t}$=0 are completely determined in this way. Exogenous statistical operators together with dynamic operators $\mathbf{U} \in \mathfrak{V}$ determine the remainder. Theoretical structures satisfying these additional conditions are models for **QM**, members of $\mathbf{M}[\mathbf{QM}\mathfrak{S}]$.

A general formulation [4.1] of **QM**-requirements is, of necessity, somewhat abstract. A more direct, intuitive insight is provided by considering some special cases [5.1]. An alternative formulation in terms of operator expectation values is sketched [4.2]. The empirical content of $\mathbf{M}[\mathbf{QM}\mathfrak{S}]$ is discussed [4.3].

4.1 General case

The most direct way to impose the desired conditions on the probability function **p** is to require that, for all $\Delta\mathbf{t} \in \mathbb{I}^{[0}$, $\mathbf{t} - (\Delta\mathbf{t} + 1) \in \mathbf{T}$,

$$\gamma_{[t-(\Delta t+1),t]} = \underset{\sim t}{o}\ \underset{\sim t}{\upsilon} \ldots \underset{\sim t-i}{o}\ \underset{\sim t-i}{\upsilon} \ldots \underset{\sim t-(\Delta t+1)}{o}\ \underset{\sim t-(\Delta t+1)}{\upsilon} \qquad [4\text{-}2]$$

there is some $|\mathbf{W}|$, a statistical operator for $\mathbb{H}$, and

$$|\mathbf{E}| \in \mathbf{SET}(\mathbf{T}, \|[\ \mathbf{L}^{s}_{o}\ ]\| \cup \mathfrak{V}) \qquad [4\text{-}3]$$

such that

$$\mathbf{Tr}\,|[\,o_{t-(\Delta t+1)}\,]|\mathbf{W}|[\,o_{t-(\hat{\Delta t}+1)}\,]| = 1 \qquad [4\text{-}4]$$

and, for all $\mathbf{i} \in \Delta\mathbf{t}$,

$$|\mathbf{E}_{t-i}| = |\mathbf{U}| \in \mathfrak{V} \Rightarrow o_{t-i}\ \upsilon_{t-i} = \mathrm{S}\{\mathrm{S}\} \quad |\mathbf{E}_{t-i}| = |\mathbf{P}| \in |[\ \mathbf{L}^{s}_{o}\ ]| \Rightarrow |[\,o_{t-i}\,]| = |\mathbf{P}| \qquad [4\text{-}5]$$

such that

$$\mathbf{p}\left(\underset{\sim t}{o} \middle| \underset{\sim t}{\upsilon} \ldots \underset{\sim t-i}{o}\ \underset{\sim t-i}{\upsilon} \ldots \underset{\sim t-\Delta t}{o}\ \underset{\sim t-\Delta t}{\upsilon}\ \underset{\sim t-(\Delta t+1)}{o}\ \underset{\sim t-(\Delta t+1)}{\upsilon} \right) = \mathbf{Tr}|\mathbf{E}_t|\mathbf{E}_{t-1}|\ldots|\mathbf{E}_{t-\Delta t}|\mathbf{W}|\hat{\mathbf{E}}_{t-\Delta t}|\ldots|\hat{\mathbf{E}}_{t-1}|\hat{\mathbf{E}}_t|.$$

[4-6]

Note that the overscript '^' in [E-4-4] pertains to the operator $|[\,o_{t-(\Delta t+1)}\,]|$ and not to the 'o' in '$o_{t-(\Delta t+1)}$'.

These are projection probabilities [E-2-46]. Recall that notation like '$\underset{\sim t}{o}$',

'$\underset{\sim t}{\upsilon}$', etc. denotes a set of result-observation paths [2.3] which are appropriate arguments for the probability function **p** while 'o_t' denotes a member of $|\mathbf{L}^s_o|$ which is an appropriate argument for [ ].

Requirement [E-4-21] assures that |**W**| assigns probability **1** to the result $\underset{\sim t-(\Delta t+1)}{o}$ of the initial observation $\underset{\sim t-(\Delta t+1)}{\upsilon}$ being in the sub-space $[o_{t-(\Delta t+1)}]$. In effect, it "initializes" the statistical operator |**W**|.

Requirement [E-4-6] determines the conditional probability of the result $\underset{\sim t}{o}$, given the observation $\underset{\sim t}{\upsilon}$, the result-observation sequence

$$\ldots \underset{\sim\ t\text{-}i}{o}\ \underset{\sim t\text{-}i}{\upsilon} \ldots \underset{\sim t\text{-}\Delta t}{\upsilon}\ \underset{\sim t\text{-}\Delta t}{o}\ \underset{\sim t-(\Delta t+1)}{o}\ \underset{\sim t-(\Delta t+1)}{\upsilon} \quad [4\text{-}7]$$

and the initial result-observable event

$$\underset{\sim t-(\Delta t+1)}{o}\ \underset{\sim t-(\Delta t+1)}{\upsilon} \quad [4\text{-}8]$$

as the value of the trace function

$$\mathbf{Tr}|\mathbf{E}_t|\mathbf{E}_{t\text{-}1}|\ldots|\mathbf{E}_{t\text{-}\Delta t}|\mathbf{W}|\hat{\mathbf{E}}_{t\text{-}\Delta t}|\ldots|\hat{\mathbf{E}}_{t\text{-}1}|\hat{\mathbf{E}}_t| \quad [4\text{-}9]$$

where

$$|\mathbf{E}_t|\mathbf{E}_{t\text{-}1}|\ldots|\mathbf{E}_{t\text{-}\Delta t}|\mathbf{W}|\hat{\mathbf{E}}_{t\text{-}\Delta t}|\ldots|\hat{\mathbf{E}}_{t\text{-}1}|\hat{\mathbf{E}}_t| \quad [4\text{-}10]$$

is the statistical operator obtained by the iterated application of *some* sequence of projection and unitary operators |**E**|, satisfying [E-4-5], on the initial statistical operator |**W**|. Note that the order of result-observable events is significant here both in determining |**W**| and in determining the order of subsequent operations on it.

Condition [E-4-5] assures that the operators in |**E**| correspond to members of $\gamma_{[t-(\Delta t+1),t]}$ in such a way that members of $\mathfrak{V}$ correspond to instances of the null result-observation event **S**{**S**} and projection operators [ ]-corresponding to their results are assigned to non-null result-observation events.

Intuitively, |**W**| represents some exogenous probability distribution over the members of the partition $\upsilon_{t-(\Delta t+1)}$ in which the result $o_{t-(\Delta t+1)}$ has probability **1** ([5.1.2.1.1]). The sequence |**E**| represents successive transformations of this probability distribution into probabilities of results $\underset{\sim t\text{-}i}{o}$ of observation events $\underset{\sim t\text{-}i}{\upsilon}$.

The expression on the right side of [E-4-6] is essentially the same as that appearing in standard expositions of quantum mechanics (Ballentine 1998, p. 46, Messiah 1961, Ch. VIII) in somewhat unfamiliar notation suggested by Nielson & Chuang (2000, Sec. 8.2.). The probability expression on the left expresses the **QM**-interpretation of the right side of as a conditional probability. This interpretation is implicit in many (but, surely not all) expositions of quantum mechanics. For **QM**, it is explicit (compare [E-4-51]).

4.2 Operator expectation values

In many discussions of quantum mechanics, what we present as the choice of a [ ]-embedding is presented as the assignment of operators in $\mathbb{H}$ to "physical" observables. The physical content of specific applications of quantum mechanics inheres in this operator assignment. For this reason it is useful to see how operator assignment appears in **QM**. A rough sketch of this is provided.

Eigenvalues of operators do not fall naturally out of the **QM** approach because results are not associated with numerical values. These can be added arbitrarily with a one-one function

$$\boldsymbol{v} \in \mathbf{ISET}(\mathbf{POT}(\mathbf{S}), \mathbb{I}^{[1,\#\mathbf{POT}(\mathbf{S})})\qquad [4\text{-}11]$$

which assigns integer "values" to all sub-sets of **S**, and thus to all results.

For [ ]-embedding [ ], $\upsilon_i \in |\mathrm{O}^s|$,

$$\left|\mathrm{O}_i^{[\ ]}\right| := |\mathrm{O}|([\ ], \upsilon_i)\qquad [4\text{-}12]$$

is the linear operator on $\mathbb{H}$ assigned to υ_i by [ ] just when the spectral representation of $\left|\mathrm{O}_i^{[\ ]}\right|$ in the orthonormal basis for $\mathbb{H}$ associated with maximal observable $\hat{\upsilon}_j \lhd^s \upsilon_i$

$$\left\{\left|{}^j\hat{o}_j\right\rangle\right\}_{j=1}^{j=n}\qquad [4\text{-}13]$$

is

$$\left[\mathrm{O}_i^{[\ ]}\right] = \left|{}^j\hat{o}_j\right\rangle \boldsymbol{v}\left(\underset{\hat{}}{\blacktriangledown}\left({}^j\hat{o}_j\right)\right)\left\langle{}^j\hat{o}_j\right|\qquad [4\text{-}14]$$

where $\underset{\hat{}}{\blacktriangledown}\left({}^j\hat{o}_j\right)$ is that unique ${}^i\hat{o}_i \in \upsilon_i$ such that ${}^i\hat{o}_i \subseteq {}^j\hat{o}_j$. The usual convention that repeated upper and lower indices '*j*' indicate summation over *j* is observed.

Only conditional operator expectation values are determined by **QM** since it determines only conditional probabilities. Thus, for operator $\left|\mathrm{O}_i^{[\ ]}\right|$ at time **t** for probability function **p** is

$$\left\langle \mathrm{O}_i^{[\ ]} \middle| \underset{\sim t}{\hat{\upsilon}}\ \underset{\sim t\text{-}1}{\hat{o}}\ \underset{\sim t\text{-}1}{\hat{\upsilon}} \right\rangle_{\mathbf{p}} = \left\langle \boldsymbol{v}(\hat{o}_t) \middle| \underset{\sim t}{\hat{\upsilon}}\ \underset{\sim t\text{-}1}{\hat{o}}\ \underset{\sim t\text{-}1}{\hat{\upsilon}} \right\rangle_{\mathbf{p}}\qquad [4\text{-}15]$$

$$= \boldsymbol{v}\left(\underset{\hat{}}{\blacktriangledown}\left({}^j\hat{o}_t\right)\right)\mathbf{p}\left({}^j\hat{o}_t \middle| \underset{\sim t}{\hat{\upsilon}}\ \underset{\sim t\text{-}1}{o}\ \underset{\sim t\text{-}1}{\upsilon}\right)\qquad (\textit{sum } j).\qquad [4\text{-}16]$$

The **QM** restriction to conditional expectation values has non-trivial consequences. The Bell-Kochen-Specker argument (Held 2008) that simultaneous assignment of values to all quantum mechanical observables can not be carried through in an obvious way for **QM** simply because **QM** operator expectation values are conditioned on observation events and results of previous observation events. This is not surprising. This conditioning is a violation of "noncontextuality" in the terminology sometimes used in discussions of this argument. Indeed, the use of conditional probabilities in **QM** might be regarded as simply a formal expression of "contextuality".

4.3 Empirical content

In the absence of constraints, the empirical claim of **QM** is that the projection probabilities in its intended applications (i.e "found in nature") can be obtained via [E-4-6] from *some*

> [ ]-embedding statistical operator |**W**| sequence of dynamic and projection operators |**E**|

satisfying [E-4-21] and [E-4-5]. Here, [ ], |W| and |**E**| play a role roughly analogous to that of the mass and force functions the familiar structuralist reconstruction of the empirical claim of classical particle mechanics (**CPM**) (Balzer, Moulines & Sneed 1987, III.3) and projection probabilities play a role analogous to that of particle paths. Clearly, in the absence of further restrictions on these constructs, *any* member of $\mathbf{M}_{\mathbf{pp}}$[**QM**$\mathfrak{S}$] can be filled out in *some* way to a member of **M**[**QM**$\mathfrak{S}$]. Just as in **CPM**, the empirical content of the "top-level" element of **QM**'s specialization net is trivial. It is only with restrictions, specifically on [ ] and the dynamic operators $\mathfrak{V}$ that the empirical content becomes non-trivial.

On an instrumentalist view [2.1.4], **QM** is simply a very general scheme for describing all possible projection probabilities. Some probabilities found in nature require [ ]'s, and (perhaps less obviously) $\mathfrak{V}$'s with surprising, counter intuitive properties. That probabilities found in nature (are thereby revealed to) have such properties is a "mysterious fact" apparently crying out for "explanation". No explanation is provided here.

5. Specialization

Specializations of **M**[**QM**$\mathfrak{S}$] may be conveniently divided into non-theoretical [5.1] and theoretical specializations. The most obvious specializations are those already mentioned [2.1] in which the underlying system structure is imported from some other theory. These are non-theoretical specializations. Other non-theoretical specializations pertain to specific kinds of sequences of result-observation events. They are analogous to particle mechanical systems with small sets of particles. Generally, theoretical specializations are identified by subsets of

$\mathfrak{V}$ [5.2.1]; among these are models in which "entangled' non-product vectors in the tensor product space for compound systems are produced by unitary transformations from $\mathfrak{V}$ acting on initial product vectors assigned to results of initial observations on the compound system [5.2.1].

5.1 Non-theoretical

It is useful to distinguish special cases of **QM**-requirements on **p** according to whether observables are all maximal [5.1.1], mixed maximal, non-maximal [5.1.2] all non-maximal and whether observations are non-null or null.

[ ]-images of results of maximal observables are essentially vectors in $\mathbb{H}$, *pure* **QM**-states; non-maximal observables are represented by a more general statistical operator |**W**|, "*mixed* **QM** -states". Non-null observations have non-trivial results; null observations have trivial results, but may result in dynamic changes in **QM**-states described by dynamic operators in $\mathfrak{V}$. The distinction between pure and mixed **QM**-states is analogous to the familiar distinction between pure and mixed quantum mechanical states.

In all cases, values of **Δt** ∈ {0, **1**} suffice to demonstrate features of **QM**-probabilities illustrating **QM** analogs to familiar features of quantum mechanical probabilities. A selection of these cases suffices for this purpose without providing exhaustive, detailed consideration of all cases.

5.1.1 Maximal observables

In the case that, for all **i** ∈ [1, **Δt** + 1], there exist $\hat{\upsilon} \in \left|\hat{\mathrm{O}}^{\mathrm{s}}\right|$, $\hat{o} \in \hat{\upsilon}$ such that

$$\underset{\sim \mathrm{t-i}}{o} \; \underset{\sim \mathrm{t-i}}{\upsilon} = \underset{\sim \mathrm{t-i}}{\hat{o}} \; \underset{\sim \mathrm{t-i}}{\hat{\upsilon}} \qquad [5\text{-}1]$$

i.e. all observables are maximal,

$$\left|\mathrm{W}\right| = \left|\hat{o}_{\mathrm{t-(\Delta t+1)}}\right\rangle\left\langle^{\mathrm{t-(\Delta t+1)}}\hat{o}\right| \qquad \left|\mathrm{E_t}\right| = \left|\hat{o}_{\mathrm{t}}\right\rangle\left\langle^{\mathrm{t}}\hat{o}\right| \qquad [5\text{-}2]$$

and

$$\mathbf{p}\left(\underset{\sim \mathrm{t}}{\hat{o}}\left|\underset{\sim \mathrm{t}}{\hat{\upsilon}} \ldots \underset{\sim \mathrm{t-i}}{\hat{o}} \; \underset{\sim \mathrm{t-i}}{\hat{\upsilon}} \ldots \underset{\sim \mathrm{t-\Delta t}}{\hat{\upsilon}} \; \underset{\sim \mathrm{t-\Delta t}}{\hat{o}} \; \underset{\sim \mathrm{t-(\Delta t+1)}}{\hat{o}} \; \underset{\sim \mathrm{t-(\Delta t+1)}}{\hat{\upsilon}}\right.\right) = \mathbf{Tr}|\mathbf{E_t}|\mathbf{E_{t\text{-}1}}|\ldots|\mathbf{E_{t\text{-}\Delta t}}|\mathbf{W}|\hat{\mathbf{E}}_{\mathbf{t\text{-}\Delta t}}|\ldots|\hat{\mathbf{E}}_{\mathbf{t\text{-}1}}|\hat{\mathbf{E}}_{\mathbf{t}}||. \qquad [5\text{-}3]$$

Intuitively, the result of an initial maximal observation **t-(Δt** + 1) is propagated by a sequence of further, non-null observations and / or dynamic operations until a final maximal observation at **t**. The special cases of sequences of *all* non-null observations [5.1.1.1] and two non-null observations separated by a sequence of null observations are considered [5.1.1.2].

5.1.1.1 Non-null

Here

$$\hat{o}_{\mathbf{i}}\hat{\upsilon}_{\mathbf{i}} \neq \mathbf{S}\{\mathbf{S}\}. \quad [5\text{-}4]$$

Thus, the [ ]-embedding suffices to determine all probabilities. Two special cases [5.1.1.1.1], [5.1.1.1.2] are considered and some implications of these cases for conditional, joint probability distributions considered [5.1.1.1.3].

5.1.1.1.1 Δt = 0

Here

$$|\mathbf{W}| = |\hat{o}_{\mathbf{t\text{-}1}}\rangle\langle^{\mathbf{t\text{-}1}}\hat{o}| \qquad |\mathbf{E}_{\mathbf{t\text{-}\Delta t}}| = |\mathbf{E}_{\mathbf{t}}| = |\hat{o}_{\mathbf{t}}\rangle\langle^{\mathbf{t}}\hat{o}|. \quad [5\text{-}5]$$

$$\begin{aligned}\mathbf{p}\left(\hat{o}_{t}\,|\hat{\upsilon}_{t}\hat{o}_{t\text{-}1}\hat{\upsilon}_{t\text{-}1}\right) &= \mathbf{Tr}|\hat{o}_{t}\rangle\langle{}_{t}\hat{o}|\hat{o}_{t\text{-}1}\rangle\langle{}_{t\text{-}1}\hat{o}|\hat{o}_{t}\rangle\langle{}_{t}\hat{o}| \\ &= \langle{}_{t}\hat{o}|\hat{o}_{t\text{-}1}\rangle\langle{}_{t\text{-}1}\hat{o}|\hat{o}_{t}\rangle \\ &= \langle{}_{t}\hat{o}|\hat{o}_{t\text{-}1}\rangle\overline{\langle{}_{t}\hat{o}|\hat{o}_{t\text{-}1}\rangle} \\ &= \left|\langle{}_{t}\hat{o}|\hat{o}_{t\text{-}1}\rangle\right|^{2}.\end{aligned} \quad [5\text{-}6]$$

The analog to [E-5-6] in quantum mechanics is sometimes called 'the Born rule' recalling the historical origin of the probabilistic interpretation of the Hilbert space formalism. It is the key feature of a probabilistic interpretation of this formalism.

5.1.1.1.2 Δt = 1

Here

$$|\mathbf{E}_{\mathbf{t\text{-}\Delta t}}| = |\mathbf{E}_{\mathbf{t\text{-}1}}| = |\hat{o}_{\mathbf{t\text{-}1}}\rangle\langle^{\mathbf{t\text{-}1}}\hat{o}| \qquad |\mathbf{E}_{\mathbf{t}}| = |\hat{o}_{\mathbf{t}}\rangle\langle^{\mathbf{t}}\hat{o}| \qquad |\mathbf{W}| = |\hat{o}_{\mathbf{t\text{-}2}}\rangle\langle^{\mathbf{t\text{-}2}}\hat{o}| \quad [5\text{-}7]$$

$$\begin{aligned}\mathbf{p}\left(\hat{o}_{t}\,|\hat{\upsilon}_{t}\hat{o}_{t\text{-}1}\hat{\upsilon}_{t\text{-}1}\hat{o}_{t\text{-}2}\hat{\upsilon}_{t\text{-}2}\right) &= \mathbf{Tr}|\hat{o}_{t}\rangle\langle{}_{t}\hat{o}|\hat{o}_{t\text{-}1}\rangle\langle{}_{t\text{-}1}\hat{o}|\hat{o}_{t\text{-}2}\rangle\langle{}_{t\text{-}2}\hat{o}|\hat{o}_{t\text{-}1}\rangle\langle{}_{t\text{-}1}\hat{o}|\hat{o}_{t}\rangle \\ &= \langle{}_{t}\hat{o}|\hat{o}_{t\text{-}1}\rangle\langle{}_{t\text{-}1}\hat{o}|\hat{o}_{t\text{-}2}\rangle\langle{}_{t\text{-}2}\hat{o}|\hat{o}_{t\text{-}1}\rangle\langle{}_{t\text{-}1}\hat{o}|\hat{o}_{t}\rangle \\ &= \left|\langle{}_{t}\hat{o}|\hat{o}_{t\text{-}1}\rangle\right|^{2}\left|\langle{}_{t\text{-}1}\hat{o}|\hat{o}_{t\text{-}2}\rangle\right|^{2} \\ &= \mathbf{p}\left(\hat{o}_{t}\,|\hat{\upsilon}_{t}\hat{o}_{t\text{-}1}\hat{\upsilon}_{t\text{-}1}\right)\mathbf{p}\left(\hat{o}_{t\text{-}1}\,|\hat{\upsilon}_{t\text{-}1}\hat{o}_{t\text{-}2}\hat{\upsilon}_{t\text{-}2}\right).\end{aligned} \quad [5\text{-}8]$$

5.1.1.1.3 Conditional joint distributions

The cases [5.1.1.1.1], [5.1.1.1.2], serve to illustrate an important point about the role of conditional joint probability distributions in **QM**.

For distinct $\hat{\upsilon}_{\mathrm{i}}$, $\hat{\upsilon}_{\mathrm{j}}$, $\hat{\upsilon}_{\mathrm{k}}$ the most obvious expression for a conditional joint distribution of the results of observations of non-commuting, maximal observables,

$$\mathbf{p}\left({}^{i}\hat{o}_{\sim i:t}\; {}^{j}\hat{o}_{\sim j:t} \middle| \hat{\upsilon}_{\sim i:t}\; \hat{\upsilon}_{\sim j:t}\; {}^{k}\hat{o}_{\sim k:t-1}\; \hat{\upsilon}_{\sim j:t-1} \right) \quad [5\text{-}9]$$

is not defined because

$$\mathbf{p}\left(\hat{\upsilon}_{\sim i:t}\; \hat{\upsilon}_{\sim j:t}\; {}^{k}\hat{o}_{\sim k:t-2}\; \hat{\upsilon}_{\sim j:t-2} \right) = 0 \quad [5\text{-}10]$$

[E-5-42]. Intuitively, **QM** simply does not consider simultaneous observation of distinct maximal observables.

The probabilities

$$\mathbf{p}\left({}^{i}\hat{o}_{\sim i:t}\; {}^{j}\hat{o}_{\sim j:t} \middle| \hat{\upsilon}_{\sim i:t}\; \hat{\upsilon}_{\sim j:t-1}\; {}^{k}\hat{o}_{\sim k:t-2}\; \hat{\upsilon}_{\sim j:t-2} \right) = \mathbf{p}\left({}^{i}\hat{o}_{\sim i:t} \middle| \hat{\upsilon}_{\sim i:t}\; {}^{j}\hat{o}_{\sim j:t-1}\; \hat{\upsilon}_{\sim j:t-1}\; {}^{k}\hat{o}_{\sim k:t-2}\; \hat{\upsilon}_{\sim j:t-2} \right) \mathbf{p}\left({}^{j}\hat{o}_{\sim j:t-1} \middle| \hat{\upsilon}_{\sim i:t}\; \hat{\upsilon}_{\sim j:t-1}\; {}^{k}\hat{o}_{\sim k:t-2}\; \hat{\upsilon}_{\sim j:t-2} \right) \quad [5\text{-}11]$$

results of $\hat{\upsilon}_i$ and $\hat{\upsilon}_j$ indexed by different values of *i*, *j* are well defined. However, **QM** [E-5-6] does *not* determine these probabilities unless one is willing to assume generally that

$$\mathbf{p}\left({}^{j}\hat{o}_{\sim j:t-1} \middle| \hat{\upsilon}_{\sim i:t}\; \hat{\upsilon}_{\sim j:t-1}\; {}^{k}\hat{o}_{\sim k:t-2}\; \hat{\upsilon}_{\sim j:t-2} \right) = \mathbf{p}\left({}^{j}\hat{o}_{\sim j:t-1} \middle| \hat{\upsilon}_{\sim j:t-1}\; {}^{k}\hat{o}_{\sim k:t-2}\; \hat{\upsilon}_{\sim j:t-2} \right). \quad [5\text{-}12]$$

Such an assumption appears quite arbitrary within the framework of **QM**.

The probabilities

$$\mathbf{p}\left({}^{i}\hat{o}_{\sim i:t} \middle| \hat{\upsilon}_{\sim i:t}\; {}^{k}\hat{o}_{\sim k:t-1}\; \hat{\upsilon}_{\sim k:t-1} \right) \qquad \mathbf{p}\left({}^{j}\hat{o}_{\sim j:t} \middle| \hat{\upsilon}_{\sim j:t}\; {}^{k}\hat{o}_{\sim k:t-1}\; \hat{\upsilon}_{\sim k:t-1} \right) \quad [5\text{-}13]$$

though determined by **QM** [E-5-6], are *not* marginal distributions derived from the joint distribution of [E-5-11]. The two conditional probabilities of [E-5-13] have different conditions. There is no reason to expect that their values, for results indexed by different values of *i*, *j*, can be derived as marginal distributions from *any* common joint distribution. The conditional probability of [E-5-11], is indeed a well defined, conditional joint probability distribution, albeit not derivable from the laws of **QM**. But the condition is different from that in either of the probabilities in [E-5-13]. It is not a joint distribution for which the probabilities in [E-5-13] are marginal distributions. For **QM**, this fact effectively blocks the analog of Wigner-Moyal (Moyal 1949, Wigner 1932) approach to attempting (unsuccessfully) a calculation of joint density functions for non-commuting observables in quantum mechanics. Probability values for results of observations non-commuting observables will always be conditional probabilities on different conditions and thus need not be derivable from a common joint distribution. That these probabilities can, not be derivable from a common joint distribution is no argument against the employment of the usual concept of mathematical probability in **QM**.

5.1.1.2 Null

Next, consider the case in which all result-observation events except initial and terminal are null, i. e. for all **i** ∈ [1, **Δt** + 1],

$$\hat{o}_{t\text{-}i}\hat{\upsilon}_{t\text{-}i} = \mathrm{S}\{\mathrm{S}\}. \quad [5\text{-}14]$$

Intuitively, after an initial preparation at **t** – (**Δt** + 1), the system is isolated and subjected to a sequence of (generally non-identity) dynamic operators from V over the interval [**t** – 1, **t** – **Δt**] and finally a observation $\hat{\upsilon}_{\sim t}$

$$\mathbf{p}\left(\hat{o}_{\sim t}\middle|\hat{\upsilon}_{\sim t}\ldots \mathrm{S}_{\sim t\text{-}i}\{\mathrm{S}\}_{\sim t\text{-}i}\,\mathrm{S}_{\sim t\text{-}\Delta t}\{\mathrm{S}\}_{\sim t\text{-}\Delta t}\,\hat{o}_{\sim t\text{-}(\Delta t+1)}\,\hat{\upsilon}_{\sim t\text{-}(\Delta t+1)}\right) =$$
$$= \mathbf{Tr}\left|\hat{o}_t\right\rangle\left\langle{}^{t}\hat{o}\right|\mathbf{U}_{t\text{-}\Delta t}\left|\ldots\right|\mathbf{U}_{t\text{-}1}\left|\mathrm{W}\right|\hat{\mathrm{U}}_{t\text{-}1}\left|\ldots\right|\hat{\mathrm{U}}_{t\text{-}\Delta t}\left|\hat{o}_t\right\rangle\left\langle{}^{t}\hat{o}\right| = \quad [5\text{-}15]$$
$$= \left|\left\langle{}^{t}\hat{o}\middle|\mathbf{v}\right\rangle\right|^2$$

where

$$\left|\mathbf{v}\right\rangle\left\langle\mathbf{v}\right| = \left|\mathbf{U}_{t\text{-}\Delta t}\right|\ldots\left|\mathbf{U}_{t\text{-}1}\right|\hat{o}_{t\text{-}(\Delta t+1)}\right\rangle\left\langle{}^{t\text{-}(\Delta t+1)}\hat{o}\right|\hat{\mathrm{U}}_{t\text{-}1}\left|\ldots\right|\hat{\mathrm{U}}_{t\text{-}\Delta t}\right|. \quad [5\text{-}16]$$

Note that there need not be any $\hat{o} \in \left|\mathbf{L}^{s}_{o}\right|$ such that

$$\left|\mathbf{v}\right\rangle\left\langle\mathbf{v}\right| = \left|[\hat{o}]\right|. \quad [5\text{-}17]$$

Intuitively, sequences of dynamic operators in 𝔙 may (but need not) carry the [ ]-image of the result of an observation event into a vector in |𝕍⟩| that is not the [ ]-image if any member of $\left|\mathbf{L}^{s}_{o}\right|$. Roughly, projection operators not in $\left|[\mathbf{L}^{s}_{o}]\right|$ may have “empirical significance” in **QM**. Whether such exist depends on the contents of 𝔙. Indeed, it may (but need not) be the case that all of |𝕍⟩| is reachable by some sequence of dynamic operators in 𝔙.

For members of **M**[**QM𝔖**], it is natural to take **QM**-*states of* **S** to be the [ ]-images members of $\left|\mathbf{L}^{s}_{o}\right|$ together with those members of $|\mathbf{L}^{\mathbb{H}}|$ which are reachable via [E-5-5] by some sequence of projection operators in $\left\|\mathbf{L}^{s}_{o}\right\|$ and dynamic operators in 𝔙.

5.1.2 Maximal, non-maximal observables

In these cases, it is convenient to expresses projection operator explicitly in terms of the distinct, non-intersecting, orthonormal bases,

$$\left\{\left|{}^{i}\hat{o}_{i}\right\rangle\right\}_{i=1}^{i=n} \qquad \left\{\left|{}^{j}\hat{o}_{j}\right\rangle\right\}_{j=1}^{j=n} \qquad \left\{\left|{}^{k}\hat{o}_{k}\right\rangle\right\}_{k=1}^{k=n}. \quad [5\text{-}18]$$

Recall [2.2.2] that

$$\left|[\boldsymbol{o}]\right| = \left|{}^{\mathrm{j}}\hat{o}_j\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right|\left(j\,\mathrm{sum},\, j\in\mathcal{J}\subset\mathbb{I}^{[1,n]};\ \mathrm{no}\ \mathbf{j}\,\mathrm{sum}\right)\Rightarrow \boldsymbol{o}=\bigcup\nolimits_{j\in\mathcal{J}}{}^{j}\hat{o}_{\mathrm{j}}\,. \qquad [5\text{-}19]$$

Only non-null observations will be considered.

5.1.2.1 Δt = 0

In the case of **Δt** = 0, there are two sub-cases

$$\underset{\sim\mathrm{t}}{\hat{o}}\ \underset{\sim\mathrm{t}}{\hat{\upsilon}}\ \underset{\sim\mathrm{t\text{-}1}}{o}\ \underset{\sim\mathrm{t\text{-}1}}{\upsilon} \qquad\qquad \underset{\sim\mathrm{t}}{o}\ \underset{\sim\mathrm{t}}{\upsilon}\ \underset{\sim\mathrm{t\text{-}1}}{\hat{o}}\ \underset{\sim\mathrm{t\text{-}1}}{\hat{\upsilon}} \qquad [5\text{-}20]$$

both of which are considered in more detail.

5.1.2.1.1 $\underset{\sim\mathrm{t}}{\hat{o}}\ \underset{\sim\mathrm{t}}{\hat{\upsilon}}\ \underset{\sim\mathrm{t\text{-}1}}{o}\ \underset{\sim\mathrm{t\text{-}1}}{\upsilon}$

QM requires [4.1] that there is some statistical operator |W| such that

$$\mathbf{Tr}|\mathrm{W}|[o_{\mathrm{t\text{-}1}}]| = 1 \qquad [5\text{-}21]$$

and

$$\mathbf{p}\left(\underset{\sim\mathrm{t}}{\hat{o}}\,\middle|\,\underset{\sim\mathrm{t}}{\hat{\upsilon}}\ \underset{\sim\mathrm{t\text{-}1}}{o}\ \underset{\sim\mathrm{t\text{-}1}}{\upsilon}\right)=\mathbf{Tr}\left|\hat{o}_{\mathrm{t}}\right\rangle\left\langle{}^{\mathrm{t}}\hat{o}\right|\mathrm{W}\left|\hat{o}_{\mathrm{t}}\right\rangle\left\langle{}^{\mathrm{t}}\hat{o}\right|. \qquad [5\text{-}22]$$

If, in the orthonormal bases of [E-5-18],

$$\begin{aligned}\left|\hat{o}_{\mathrm{t}}\right\rangle\left\langle{}^{\mathrm{t}}\hat{o}\right| &= \left|{}^{\mathrm{i}}\hat{o}_i\right\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|(\mathrm{no\ sum})\\ \left|[\boldsymbol{o}_{\mathrm{t\text{-}1}}]\right| &= \left|{}^{\mathrm{j}}\hat{o}_j\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right|\left(j\,\mathrm{sum},\, j\in\mathcal{J}\subset\mathbb{I}^{[1,n]};\ \mathrm{no}\ \mathbf{j}\,\mathrm{sum}\right)\end{aligned} \qquad [5\text{-}23]$$

then there exist

$${}^{j}\mathrm{W}_j\in\mathbb{I}^{[0,1]},\quad {}^{j}\mathbf{W}_j=1\quad\left(j\,\mathrm{sum},\, j\in\mathcal{J}\subset\mathbb{I}^{[1,n]}\right) \qquad [5\text{-}24]$$

such that

$$\left|\mathrm{W}\right|=\left|{}^{\mathrm{j}}\hat{o}_j\right\rangle{}^{i}\mathrm{W}_i\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right|\ \left(j\,\mathrm{sum},\, j\in\mathcal{J}\subset\mathbb{I}^{[1,n]}\right) \qquad [5\text{-}25]$$

and

$$\begin{aligned}\mathbf{p}\left(\underset{\sim\mathrm{t}}{\hat{o}}\,\middle|\,\underset{\sim\mathrm{t}}{\hat{\upsilon}}\ \underset{\sim\mathrm{t\text{-}1}}{o}\ \underset{\sim\mathrm{t\text{-}1}}{\upsilon}\right)&=\mathbf{Tr}\left|{}^{\mathrm{i}}\hat{o}_i\right\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|{}^{\mathrm{j}}\hat{o}_j\rangle{}^{j}\mathrm{W}_j\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right|{}^{\mathrm{i}}\hat{o}_i\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|\quad\left(j\,\mathrm{sum},\, j\in\mathcal{J}\subset\mathbb{I}^{[1,n]}\right)\\ &={}^{j}\mathrm{W}_j\left|\left\langle{}^{j}\hat{o}_{\mathrm{j}}\middle|{}^{\mathrm{i}}\hat{o}_i\right\rangle\right|^2\\ &={}^{j}\mathrm{W}_j\mathbf{p}\left({}^{i}\underset{\sim\mathrm{i}}{\hat{o}}\,\middle|\,\underset{\sim\mathrm{i}}{\hat{\upsilon}}\ {}^{j}\underset{\sim\mathrm{j}}{\hat{o}}\ \underset{\sim\mathrm{j}}{\hat{\upsilon}}\right)\end{aligned} \qquad [5\text{-}26]$$

Intuitively, the ${}^{j}\mathbf{W}_j$ are a probability distribution over the vectors $\left|{}^{\mathrm{j}}\hat{o}_j\right\rangle$ which span the subspace $[o_{\mathrm{t}-1}]$. ${}^{j}\mathrm{W}_j$ is the probability that the **QM**-state $\left|{}^{\mathrm{j}}\hat{o}_j\right\rangle$ represents the result of observation $\underset{\sim\mathrm{t\text{-}1}}{\upsilon}$. This probability is not determined by **QM**.

5.1.2.1.2 $\underset{\sim t}{o}\ \underset{\sim t}{\upsilon}\ \underset{\sim t\text{-}1}{\hat{o}}\ \underset{\sim t\text{-}1}{\hat{\upsilon}}$

If, in the orthonormal bases of [E-5-18],

$$\begin{aligned} &\left|[o_{\mathrm{t}}]\right| = \left|{}^{\mathrm{i}}\hat{o}_i\right\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|\left(i\text{ sum}, i\in \mathcal{I}\subset \mathbb{I}^{[1,\mathrm{n}]}; \text{no } \mathbf{i} \text{ sum}\right)\\ &\left|\hat{o}_{\mathrm{t\text{-}1}}\right\rangle\left\langle{}^{\mathrm{t\text{-}1}}\hat{o}\right| = \left|{}^{\mathrm{j}}\hat{o}_j\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right|\left(\text{no sum}\right)\\ &\left|\mathbf{W}\right| = \left|{}^{\mathrm{j}}\hat{o}_j\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right|\left(\text{no sum}\right)\end{aligned} \qquad [5\text{-}27]$$

$$\begin{aligned} &\mathbf{p}\left(\underset{\sim t}{o}\middle|\underset{\sim t}{\upsilon}\ \underset{\sim t\text{-}1}{\hat{o}}\ \underset{\sim t\text{-}1}{\hat{\upsilon}}\right) = \mathbf{Tr}\left(\left|{}^{\mathrm{i}}\hat{o}_i\right\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|\right)\left|{}^{\mathrm{j}}\hat{o}_j\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right|\left(\left|{}^{\mathrm{i}}\hat{o}_i\right\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|\right) \quad \left(i\text{ sum}, i\in \mathcal{I}\subset \mathbb{I}^{[1,\mathrm{n}]}; \text{no } \mathbf{i}, \mathbf{j}, j \text{ sum}\right)\\ &= \mathbf{Tr}\left|{}^{\mathrm{i}}\hat{o}_i\right\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|{}^{\mathrm{j}}\hat{o}_j\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right|{}^{\mathrm{i}}\hat{o}_i\right\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|\\ &= \textstyle\sum_{i\in\mathcal{I}}\left\langle{}^{i}\hat{o}_{\mathrm{i}}\middle|{}^{\mathrm{i}}\hat{o}_i\right\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\middle|{}^{\mathrm{j}}\hat{o}_j\right\rangle\\ &= \textstyle\sum_{i\in\mathcal{I}}\left|\left\langle{}^{j}\hat{o}_{\mathrm{j}}\middle|{}^{\mathrm{i}}\hat{o}_i\right\rangle\right|^2\\ &= \textstyle\sum_{i\in\mathcal{I}}\mathbf{p}\left({}^{i}\underset{\sim i}{\hat{o}}\middle|\underset{\sim i}{\hat{\upsilon}}\ {}^{j}\underset{\sim j}{\hat{o}}\ \underset{\sim j}{\hat{\upsilon}}\right)\end{aligned} \qquad [5\text{-}28]$$

5.1.2.2 Δ**t** = 1

In the case of Δ**t** = 1, there are **4** sub-cases

$\underset{\sim t}{\hat{o}}\ \underset{\sim t}{\hat{\upsilon}}\ \underset{\sim t\text{-}1}{o}\ \underset{\sim t\text{-}1}{\upsilon}\ \underset{\sim t\text{-}2}{\hat{o}}\ \underset{\sim t\text{-}2}{\hat{\upsilon}}$ $\quad$ $\underset{\sim t}{o}\ \underset{\sim t}{\upsilon}\ \underset{\sim t\text{-}1}{\hat{o}}\ \underset{\sim t\text{-}1}{\hat{\upsilon}}\ \underset{\sim t\text{-}2}{\hat{o}}\ \underset{\sim t\text{-}2}{\hat{\upsilon}}$ $\quad$ $\underset{\sim t}{\hat{o}}\ \underset{\sim t}{\hat{\upsilon}}\ \underset{\sim t\text{-}1}{\hat{o}}\ \underset{\sim t\text{-}1}{\hat{\upsilon}}\ \underset{\sim t\text{-}2}{o}\ \underset{\sim t\text{-}2}{\upsilon}$ [5-29]

$\underset{\sim t}{o}\ \underset{\sim t}{\upsilon}\ \underset{\sim t\text{-}1}{o}\ \underset{\sim t\text{-}1}{\upsilon}\ \underset{\sim t\text{-}2}{\hat{o}}\ \underset{\sim t\text{-}2}{\hat{\upsilon}}$.

Of these, only $\underset{\sim t}{\hat{o}}\ \underset{\sim t}{\hat{\upsilon}}\ \underset{\sim t\text{-}1}{o}\ \underset{\sim t\text{-}1}{\upsilon}\ \underset{\sim t\text{-}2}{\hat{o}}\ \underset{\sim t\text{-}2}{\hat{\upsilon}}$ will be considered in detail. It illustrates "interference" for **QM** probabilities. First, the abstract mathematics is considered [5.1.2.2.1] then an interpretation in terms of slit interference is provided.

5.1.2.2.1 Mathematics

If

$$\hat{\upsilon}_{\mathrm{j}} \lhd^{\mathrm{s}} \upsilon_{\mathrm{j}} = \upsilon_{\mathrm{t\text{-}1}} \qquad [5\text{-}30]$$

so that

$$\begin{aligned} &[o_{\mathrm{t}}] = \left|{}^{\mathrm{i}}\hat{o}_i\right\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|\\ &\left|[o_{\mathrm{t\text{-}1}}]\right| = \left|{}^{\mathrm{j}}\hat{o}_j\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right|\left(j\text{ sum}, j\in \mathcal{J}\subset \mathbb{I}^{[1,\mathrm{n}]}\right)\\ &\left|\mathbf{W}\right| = \left|{}^{\mathrm{k}}\hat{o}_k\right\rangle\left\langle{}^{k}\hat{o}_{\mathrm{k}}\right|\end{aligned} \qquad [5\text{-}31]$$

then

$$
\begin{aligned}
&\mathbf{p}\left(\hat{o}_{\sim t}\middle|\hat{\upsilon}_{\sim t}\, o_{\sim t\text{-}1}\, \upsilon_{\sim t\text{-}1}\, \hat{o}_{\sim t\text{-}2}\, \hat{\upsilon}_{\sim t\text{-}2}\right)=\mathbf{p}\left({}^{i}\hat{o}_{\sim i}\middle|\hat{\upsilon}_{\sim i}\, o_{\sim j}\, \upsilon_{\sim j}\, {}^{k}\hat{o}_{\sim k}\, \hat{\upsilon}_{\sim k}\right)\\
&=\mathbf{Tr}\left|{}^{\mathrm{i}}\hat{o}_{i}\right\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|\left(\left|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right|\right)\left|{}^{\mathrm{k}}\hat{o}_{k}\right\rangle\left\langle{}^{k}\hat{o}_{\mathrm{k}}\right|\left(\left|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right|\right)\left|{}^{\mathrm{i}}\hat{o}_{i}\right\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|\\
&=\mathbf{Tr}\left|{}^{\mathrm{i}}\hat{o}_{i}\right\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|\left(\left|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right|{}^{\mathrm{k}}\hat{o}_{k}\right\rangle\left\langle{}^{k}\hat{o}_{\mathrm{k}}\right|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right|\right)\left|{}^{\mathrm{i}}\hat{o}_{i}\right\rangle\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|\\
&=\mathbf{Tr}\left|{}^{\mathrm{i}}\hat{o}_{i}\right\rangle\left(\left\langle{}^{i}\hat{o}_{\mathrm{i}}\middle|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\middle|{}^{\mathrm{k}}\hat{o}_{k}\right\rangle\right)\left(\left\langle{}^{k}\hat{o}_{\mathrm{k}}\middle|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\middle|{}^{\mathrm{i}}\hat{o}_{i}\right\rangle\right)\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right|\\
&=\mathbf{Tr}\left|{}^{\mathrm{i}}\hat{o}_{i}\right\rangle\left(\left\langle{}^{i}\hat{o}_{\mathrm{i}}\middle|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\middle|{}^{\mathrm{k}}\hat{o}_{k}\right\rangle\right)\overline{\left(\left\langle{}^{k}\hat{o}_{\mathrm{k}}\middle|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\middle|{}^{\mathrm{i}}\hat{o}_{i}\right\rangle\right)}\left\langle{}^{i}\hat{o}_{\mathrm{i}}\right| \qquad [5\text{-}32]\\
&=\left|\sum\nolimits_{j\in\mathcal{J}}\left\langle{}^{i}\hat{o}_{\mathrm{i}}\middle|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\middle|{}^{\mathrm{k}}\hat{o}_{k}\right\rangle\right|^{2}\\
&=\sum\nolimits_{j\in\mathcal{J}}\left|\left\langle{}^{i}\hat{o}_{\mathrm{i}}\middle|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\middle|{}^{\mathrm{k}}\hat{o}_{k}\right\rangle\right|^{2}+\mathrm{IF}\\
&=\sum\nolimits_{j\in\mathcal{J}}\left|\left\langle{}^{i}\hat{o}_{\mathrm{i}}\middle|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\right|^{2}\left|\left\langle{}^{j}\hat{o}_{\mathrm{j}}\middle|{}^{\mathrm{k}}\hat{o}_{k}\right\rangle\right|^{2}+\mathrm{IF}\\
&=\sum\nolimits_{j\in\mathcal{J}}\mathbf{p}\left({}^{i}\hat{o}_{\sim i}\middle|\hat{\upsilon}_{\sim i}\,{}^{j}\hat{o}_{\sim j}\,\hat{\upsilon}_{\sim j}\right)\mathbf{p}\left({}^{j}\hat{o}_{\sim j}\middle|\hat{\upsilon}_{\sim j}\,{}^{k}\hat{o}_{\sim k}\,\hat{\upsilon}_{\sim k}\right)+\mathrm{IF}
\end{aligned}
$$

where

$$
\mathrm{IF}=2\sum\nolimits_{j',\,j\in\mathcal{J}}\left|\left\langle{}^{i}\hat{o}_{\mathrm{i}}\middle|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\middle|{}^{\mathrm{k}}\hat{o}_{k}\right\rangle\right|\left|\left\langle{}^{i}\hat{o}_{\mathrm{i}}\middle|{}^{\mathrm{j}}\hat{o}_{j'}\right\rangle\left\langle{}^{j'}\hat{o}_{\mathrm{j}}\middle|{}^{\mathrm{k}}\hat{o}_{k}\right\rangle\right|cos(\theta-\theta') \qquad [5\text{-}33]
$$

and θ pertains to the polar representation of the complex inner products. Traditionally, the summands in IF are called 'interferfence terms'.

5.1.2.2.2 Interference

The mathematics of [5.1.2.2.1] can be interpreted to provide an analog of the familiar quantum mechanical treatment of "slit interference". The treatment here closely follows that of Feynman in (Feynman, Leighton & Sands 1965, Sec. 3-1). Roughly, it attempts to show that essential features of this discussion can be reproduced within the conceptual framework of **QM**. The further discussion of Sec. 3-2 can also be rendered in the **QM** framework considering a compound electron-photon system, but this is not described here.

In Feynman's terms, make the intuitive identifications:

$$
\hat{\upsilon}_{\mathrm{k}}\sim\text{source}\qquad\left|{}^{\mathrm{k}}\hat{o}_{k}\right\rangle\sim\text{source state }|\mathbf{s}\rangle\qquad\hat{\upsilon}_{\mathrm{i}}\sim\text{detector}\qquad\left|{}^{\mathrm{i}}\hat{o}_{i}\right\rangle\sim\text{detector state }|\mathbf{x}_{i}\rangle \qquad [5\text{-}34]
$$

$$
\hat{\upsilon}_{\mathrm{j}}\sim\text{wall}\qquad\left|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\sim\text{wall states }|j\rangle. \qquad [5\text{-}35]
$$

Feynman considers examples only of **1** and **2** state "walls" the states of which are $|\mathbf{1}\rangle$, $|\mathbf{2}\rangle$.

Intuitively, in Feynman's terminology, the source system produces a beam of particles in state |**s**⟩; the wall system is a configuration of "slits" in a "wall" through which the beam passes; the detector system is a "screen" which collects the particles. In the **QM** treatment, there is only one system **S**, the particle, and three different observables for this system. The probabilities [E-5-32] for a fixed value of $\underset{\sim\mathrm{i}}{\hat{\upsilon}}\,\underset{\sim\mathrm{j}}{o}\,\underset{\sim\mathrm{j}}{\upsilon}\,{}^{k}\underset{\sim\mathrm{k}}{\hat{o}}\,\underset{\sim\mathrm{k}}{\hat{\upsilon}}$ and different values of $\boldsymbol{i}$ in ${}^{i}\underset{\sim\mathrm{i}}{\hat{o}}$ provide a probability distribution of detector-observable results. One may envision this distribution displayed vertically on a "screen" in the manner of the diagrams in Feynman's discussion. These probabilities correspond to relative numbers of particles collected at a specific position on the screen.

In the **QM** treatment, the source beam is a maximal result-observation event ${}^{k}\underset{\sim\mathrm{k}}{\hat{o}}\,\underset{\sim\mathrm{k}}{\hat{\upsilon}}$, represented by it's [ ]-image, the **QM** state $\left|{}^{\mathrm{k}}\hat{o}_{k}\right\rangle$. The passage of the beam through the wall is the nonmaximal result observation event $\underset{\sim\mathrm{j}}{o}\,\underset{\sim\mathrm{j}}{\upsilon}$, represented by

$$\left|[o_{\mathrm{j}}]\right| = \left|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right| \left(j\ \mathrm{sum},\ j \in \mathcal{J} \subset \mathbb{I}^{[1,\mathrm{n}]}\right) \qquad o_{\mathrm{j}} = \bigcup_{j\in\mathcal{J}} {}^{j}\hat{o}_{\mathrm{j}}\,. \qquad [5\text{-}36]$$

The wall itself may be viewed as the non-maximal observable υ_{j}. The arrival of the beam at the detector is the maximal result-observation event ${}^{i}\underset{\sim\mathrm{i}}{\hat{o}}\,\underset{\sim\mathrm{i}}{\hat{\upsilon}}$, represented by $\left|{}^{\mathrm{i}}\hat{o}_{i}\right\rangle$.

Walls may be viewed more generally as non-maximal observables υ_{j} such that

$$\hat{\upsilon}_{\mathrm{j}} \lhd^{\mathrm{s}} \upsilon_{\mathrm{j}} \qquad [5\text{-}37]$$

where $\hat{\upsilon}_{\mathrm{j}}$ is any maximal observable distinct from, and non-intersecting with, $\hat{\upsilon}_{\mathrm{i}}$ and $\hat{\upsilon}_{\mathbf{k}}$. The null observable {S} may be viewed as an **n**-slit wall

$$\left|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right| = \left|\mathbf{I}\right|\left(j\ \mathrm{sum},\ j \in \mathbb{I}^{[1,\mathrm{n}]}\right) \qquad [5\text{-}38]$$

each elementary projection operator in the sum being a slit. Walls with **m** < **n** slits are produced by closing slits in the {S} wall to obtain

$$\left|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right| \quad \left(j\ \mathrm{sum},\ j \in \mathcal{J} \subset \mathbb{I}^{[1,\mathrm{n}]}\right) \qquad [5\text{-}39]$$

where the elementary projection operators in the sum are the remaining open slits. A **1**-slit wall is

$$\left|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right| \qquad \text{(no sum).} \qquad [5\text{-}40]$$

A **2**-slit wall

$$\left|{}^{\mathrm{j}}\hat{o}_{j}\right\rangle\left\langle{}^{j}\hat{o}_{\mathrm{j}}\right| + \left|{}^{\mathrm{j}}\hat{o}_{j'}\right\rangle\left\langle{}^{j'}\hat{o}_{\mathrm{j}}\right| \quad j \neq j' \qquad [5\text{-}41]$$

is obtained from this **1**-slit wall by opening another slit corresponding to $\hat{\upsilon}_{\mathrm{j}}$ result ${}^{\mathrm{j}}\hat{o}_{j'}$.

A wall may be viewed as a selection device for results of maximal observable $\hat{\upsilon}_{j}$. A **1**-slit wall selects a single result; a **2**-slit wall selects a set of two results, etc. The **n**-slit wall selects all results, i. e. no results. Both source and detector maximal observables may be viewed as **1**-slit walls though this is not usually regarded as relevant to discussions of interference. Walls with **1** [5.1.2.2.2.1] and **2** slits [5.1.2.2.2.2] are considered in more detail.

5.1.2.2.2.1 1-Slit $\#\mathcal{J} = 1$

If $\#\mathcal{J} = 1$, the wall has "one slit".

$$\left| {}^{j}\hat{o}_{1} \right\rangle \sim \text{wall state } |1\rangle. \qquad [5\text{-}42]$$

This case is identical with [5.1.1.1.2] above where, now in more explicit notation,

$$\begin{aligned} \mathbf{p}\left(\hat{o}_{t} \,\middle|\, \hat{\upsilon}_{t}\hat{o}_{t\text{-}1}\hat{\upsilon}_{t\text{-}1}\hat{o}_{t\text{-}2}\hat{\upsilon}_{t\text{-}2}\right) &= \mathbf{p}\left({}^{i}\hat{o}_{\sim i} \,\middle|\, \hat{\upsilon}_{\sim i}\, {}_{1}\hat{o}_{\sim j}\, \hat{\upsilon}_{\sim j}\, {}^{k}\hat{o}_{\sim k}\, \hat{\upsilon}_{\sim k}\right) \\ &= \mathbf{p}\left({}^{i}\hat{o}_{\sim i} \,\middle|\, \hat{\upsilon}_{\sim i}\, {}_{1}\hat{o}_{\sim j}\, \hat{\upsilon}_{\sim j}\right) \mathbf{p}\left({}_{1}\hat{o}_{\sim j} \,\middle|\, \hat{\upsilon}_{\sim j}\, {}^{k}\hat{o}_{\sim k}\, \hat{\upsilon}_{\sim k}\right). \end{aligned} \qquad [5\text{-}43]$$

5.1.2.2.2.2 2-Slit $\#\mathcal{J} = 2$

If $\#\mathcal{J} = 2$, the wall has "two slits".

$$\left| {}^{j}\hat{o}_{1} \right\rangle, \left| {}^{j}\hat{o}_{2} \right\rangle \sim \text{wall states } |1\rangle, |2\rangle \qquad [5\text{-}44]$$

$$\begin{aligned} &[o_{t}] = \left| {}^{i}\hat{o}_{i} \right\rangle \left\langle {}^{i}\hat{o}_{i} \right| \\ &\left|[o_{t\text{-}1}]\right| = \left| {}^{j}\hat{o}_{1} \right\rangle \left\langle {}^{1}\hat{o}_{j} \right| + \left| {}^{j}\hat{o}_{2} \right\rangle \left\langle {}^{2}\hat{o}_{j} \right| \\ &o_{t\text{-}1} = {}^{1}\hat{o}_{j} \cup {}^{2}\hat{o}_{j} \\ &|\mathrm{W}| = \left| {}^{k}\hat{o}_{k} \right\rangle \left\langle {}^{k}\hat{o}_{k} \right| \end{aligned} \qquad [5\text{-}45]$$

and [E-5-32] reduces to

$$\begin{aligned} \mathbf{p}\left({}^{i}\hat{o}_{\sim i} \,\middle|\, \hat{\upsilon}_{\sim i}\, o_{\sim j}\, \upsilon_{\sim j}\, {}^{k}\hat{o}_{\sim k}\, \hat{\upsilon}_{\sim k}\right) &= \mathbf{p}\left({}^{i}\hat{o}_{\sim i} \,\middle|\, \hat{\upsilon}_{\sim i}\, {}_{1}\hat{o}_{\sim j}\, \hat{\upsilon}_{\sim j}\right) \mathbf{p}\left({}_{1}\hat{o}_{\sim j} \,\middle|\, \hat{\upsilon}_{\sim j}\, {}^{k}\hat{o}_{\sim k}\, \hat{\upsilon}_{\sim k}\right) + \\ &+ \mathbf{p}\left({}^{i}\hat{o}_{\sim i} \,\middle|\, \hat{\upsilon}_{\sim i}\, {}_{2}\hat{o}_{\sim j}\, \hat{\upsilon}_{\sim j}\right) \mathbf{p}\left({}_{2}\hat{o}_{\sim j} \,\middle|\, \upsilon_{\sim j}\, {}^{k}\hat{o}_{\sim k}\, \hat{\upsilon}_{\sim k}\right) + \mathrm{IF}. \end{aligned} \qquad [5\text{-}46]$$

Thus, probabilities $\mathbf{p}\left({}^{i}\hat{o}_{\sim i} \,\middle|\, \hat{\upsilon}_{\sim i}\, o_{\sim j}\, \upsilon_{\sim j}\, {}^{k}\hat{o}_{\sim k}\, \hat{\upsilon}_{\sim k}\right)$ at the detector are not the sum of probabilities for the **1**- slit observables. There are "interference terms" IF.

Within the austere conceptual apparatus of **QM**, this result is not particularly surprising or counter intuitive. There is no good reason to expect that the probabilities at the detector would simply be the sum of the conditional probabilities segments of the separate "paths" from the source to the detector calculated in the manner of [5.1.1.1.2]. But, there are some bad reasons. This expectation might arise from (perhaps implicitly) taking the probabilities in question to be unconditional "path probabilities"

$$\begin{aligned}\mathbf{p}\left({}^{i}\underset{\sim i}{\hat{o}}\,\underset{\sim i}{\hat{v}}\,\underset{\sim j}{o}\,\underset{\sim j}{v}\,{}^{k}\underset{\sim k}{\hat{o}}\,\underset{\sim k}{\hat{v}}\right)&=\mathbf{p}\left({}^{i}\underset{\sim i}{\hat{o}}\,\underset{\sim i}{\hat{v}}\left({}^{1}\underset{\sim j}{\hat{o}}\cup{}^{2}\underset{\sim j}{\hat{o}}\right)\underset{\sim j}{v}\,{}^{k}\underset{\sim k}{\hat{o}}\,\underset{\sim k}{\hat{v}}\right)=\\&=\mathbf{p}\left({}^{i}\underset{\sim i}{\hat{o}}\,\underset{\sim i}{\hat{v}}\,{}^{1}\underset{\sim j}{\hat{o}}\,\underset{\sim j}{v}\,{}^{k}\underset{\sim k}{\hat{o}}\,\underset{\sim k}{\hat{v}}\cup{}^{i}\underset{\sim i}{\hat{o}}\,\underset{\sim i}{\hat{v}}\,{}^{2}\underset{\sim j}{\hat{o}}\,\underset{\sim j}{v}\,{}^{k}\underset{\sim k}{\hat{o}}\,\underset{\sim k}{\hat{v}}\right)=\\&=\mathbf{p}\left({}^{i}\underset{\sim i}{\hat{o}}\,\underset{\sim i}{\hat{v}}\,{}^{1}\underset{\sim j}{\hat{o}}\,\underset{\sim j}{v}\,{}^{k}\underset{\sim k}{\hat{o}}\,\underset{\sim k}{\hat{v}}\right)+\mathbf{p}\left({}^{i}\underset{\sim i}{\hat{o}}\,\underset{\sim i}{\hat{v}}\,{}^{2}\underset{\sim j}{\hat{o}}\,\underset{\sim j}{v}\,{}^{k}\underset{\sim k}{\hat{o}}\,\underset{\sim k}{\hat{v}}\right).\end{aligned}\qquad[5\text{-}47]$$

But, these are not among the probabilities determined by **QM**. Probabilities actually determined by **QM** yield

$$\begin{aligned}\mathbf{p}\left({}^{i}\underset{\sim i}{\hat{o}}\,\middle|\,\underset{\sim i}{\hat{v}}\,\underset{\sim j}{o}\,\underset{\sim j}{v}\,{}^{k}\underset{\sim k}{\hat{o}}\,\underset{\sim k}{\hat{v}}\right)&=\mathbf{p}\left({}^{i}\underset{\sim i}{\hat{o}}\,\middle|\,\underset{\sim i}{\hat{v}}\left({}^{1}\underset{\sim j}{\hat{o}}\cup{}^{2}\underset{\sim j}{\hat{o}}\right)\underset{\sim j}{v}\,{}^{k}\underset{\sim k}{\hat{o}}\,\underset{\sim k}{\hat{v}}\right)=\\&=\mathbf{p}\left({}^{i}\underset{\sim i}{\hat{o}}\,\middle|\,\underset{\sim i}{\hat{v}}\,{}^{1}\underset{\sim j}{\hat{o}}\,\underset{\sim j}{v}\,{}^{k}\underset{\sim k}{\hat{o}}\,\underset{\sim k}{\hat{v}}\cup{}^{i}\underset{\sim i}{\hat{o}}\,\underset{\sim i}{\hat{v}}\,{}^{2}\underset{\sim j}{\hat{o}}\,\underset{\sim j}{v}\,{}^{k}\underset{\sim k}{\hat{o}}\,\underset{\sim k}{\hat{v}}\right).\end{aligned}\qquad[5\text{-}48]$$

On might be tempted to identify the expression on the right of the last line above with

$$\mathbf{p}\left({}^{i}\underset{\sim i}{\hat{o}}\,\middle|\,\underset{\sim i}{\hat{v}}\,{}^{1}\underset{\sim j}{\hat{o}}\,\underset{\sim j}{v}\,{}^{k}\underset{\sim k}{\hat{o}}\,\underset{\sim k}{\hat{v}}\right)+\mathbf{p}\left({}^{i}\underset{\sim i}{\hat{o}}\,\middle|\,\underset{\sim i}{\hat{v}}\,{}^{2}\underset{\sim j}{\hat{o}}\,\underset{\sim j}{v}\,{}^{k}\underset{\sim k}{\hat{o}}\,\underset{\sim k}{\hat{v}}\right).\qquad[5\text{-}49]$$

But, this would be a mistake. This identification is not a consequence of the usual axioms of probability theory. Further, the conditional probabilities appearing as summands above are not among those determined by **QM**.

That the quantum mechanical result analogous to [E-5-46] is surprising and counter intuitive appears to be a consequence of features of the underlying theory about $\mathbb{S}$ [2.1.4]. In particular, quantum mechanical systems have spatial concepts which carry with them certain pre-theoretic, intuitive expectations. That these expectations are not met is (was historically) surprising.

5.2 Theoretical

Special cases of unitary operators in two-dimensional observable structures [5.2.1] and entanglement models [5.2.2] are considered.

5.2.1 Two dimensional $\mathfrak{V}$

Consider a **2**-dimensional observable structure $\mathbf{O}^{s}$, such that

$$\left|\mathbf{O}^{s}\right| = \{\hat{\upsilon}_1, \hat{\upsilon}_2, \mathrm{S}, \boldsymbol{\Lambda}\} \qquad \hat{\upsilon}_1 = \left\{{}^1\hat{o}_1, {}^2\hat{o}_1\right\} \qquad \hat{\upsilon}_2 = \left\{{}^1\hat{o}_2, {}^2\hat{o}_2\right\} \tag{5-50}$$

and a [ ]-embedding yielding the orthonormal basies

$$\mathbf{B}^1_\perp = \left\{\left|{}_1\hat{o}_1\right\rangle, \left|{}_1\hat{o}_2\right\rangle\right\} \qquad \mathbf{B}^2_\perp = \left\{\left|{}_2\hat{o}_1\right\rangle, \left|{}_2\hat{o}_2\right\rangle\right\}. \tag{5-51}$$

Let $\mathfrak{U}$ be the set of all unitary operators on a **2**-dimensional $\mathbb{H}$.

Consider some special cases of |**U**| ∈ $\mathfrak{U}$, exhibiting their matrix representations (denoted generally by '$\mathbf{U}^i$' without '| |') in the orthonormal basis $\mathbf{B}^i_\perp$:

|**X**|

$$\mathbf{X}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \tag{5-52}$$

$$\left|\mathbf{X}\right| {}^1\hat{o}_1\rangle = \left|{}^1\hat{o}_2\right\rangle \qquad \mathbf{X}\left|{}^1\hat{o}_2\right\rangle = \left|{}^1\hat{o}_1\right\rangle \tag{5-53}$$

|**Z**|

$$\mathbf{Z}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} \tag{5-54}$$

$$\left|\mathbf{X}\right| {}^1\hat{o}_1\rangle = \left|{}^1\hat{o}_1\right\rangle \qquad \mathbf{X}\left|{}^1\hat{o}_2\right\rangle = -\left|{}^1\hat{o}_1\right\rangle \tag{5-55}$$

|**H**|

$$\mathbf{H}^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha+\beta \\ \alpha-\beta \end{pmatrix} \tag{5-56}$$

$$\left|\mathbf{H}\right| {}^1\hat{o}_1\rangle = \left(\left|{}^1\hat{o}_1\right\rangle + \left|{}^1\hat{o}_2\right\rangle\right)/\sqrt{2} \quad \left|\mathbf{H}\right| {}^1\hat{o}_2\rangle = \left(\left|{}^1\hat{o}_1\right\rangle - \left|{}^1\hat{o}_2\right\rangle\right)/\sqrt{2} \tag{5-57}$$

More generally, let

$\mathfrak{U}^1$ = {|U|^1 | ∃|U| ∈ $\mathfrak{U}$ ∋ $\mathbf{U}^1$ is the matrix representation of |**U**| in $\mathbf{B}^1_\perp$ }. [5-58]

Abusing notation, consider

$$\mathbf{U}^1 \in \mathbf{SET}(|\mathbb{R}|^4, \mathfrak{U}^1) \tag{5-59}$$

such that, for all α, β, γ, δ ∈ $|\mathbb{R}|^4$,

$$\mathbf{U}^1(\alpha,\beta,\gamma,\delta) = \begin{pmatrix} e^{-i\beta\beta/} & 0 \\ 0 & e^{i\beta\beta/} \end{pmatrix}\begin{pmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix}\begin{pmatrix} e^{-i\delta\delta/} & 0 \\ 0 & e^{i\delta\delta/} \end{pmatrix}. \tag{5-60}$$

It can be shown that (Nielson & Chuang 2000, p. 20)

$$\mathbf{U}^1 \in \mathbf{SET}(|\mathbb{R}|^4, \mathfrak{U}^1). \quad [5\text{-}61]$$

Define an equivalence relation $\underset{\mathrm{U}}{\approx}$ on $|\mathbb{R}|^4$ by

$$\rho \underset{\mathrm{U}}{\approx} \rho' \Leftrightarrow \mathrm{U}^1(\rho) = \mathrm{U}^1(\rho'). \quad [5\text{-}62]$$

Then, sub-sets of $|\mathbb{R}|^4 / \underset{\mathrm{U}}{\approx}$ determine theoretical specializations of the non-theoretical specialization of **QM** to **2**-dimensional observable structure.

5.2.2 Entanglement

Entanglement models for **QM** are roughly those in which "entangled" non-product vectors in the tensor product space representing the **QM** -state for the compound system have the following property. Projection probabilities for results of observations of component system observables for each of the compound systems are "correlated" in such a way that the probability of a component system observable result for one component, given that of another, is either **1** or **0**. Entangled vectors do not represent the result of any observation on the compound system. Rather, they are produced by unitary transformations from $\mathfrak{V}$ acting on initial product vectors assigned to results of initial observations on the compound system.

More precisely, an *entanglement model* for **QM** is an

$$\langle \mathbf{S}^{\mathrm{I,II}}, \mathbf{T}, \mathbf{p}, \mathbb{H}^{\mathrm{I}\otimes\mathrm{II}}, [\;]^{\mathrm{I}\otimes\mathrm{II}}, \mathfrak{V}^{\mathrm{I}\otimes\mathrm{II}} \rangle \in \mathbf{M}[\mathbf{QM}\mathfrak{S}] \quad [5\text{-}63]$$

such that there exist $\Delta\mathbf{t} \in \mathbb{I}^{[0}$, $\mathbf{t} - (\Delta\mathbf{t}{+}1) \in \mathbf{T}$,

$$\begin{gathered} {}^{\mathrm{i}}o^{\mathrm{I}\circ\mathrm{II}}_{\mathrm{I}}{}_{\mathrm{t}}\hat{\upsilon}^{\mathrm{I}\circ\mathrm{II}}_{\mathrm{I}}{}_{\mathrm{t}}\,{}^{\mathrm{j}}o^{\mathrm{I}\circ\mathrm{II}}_{\mathrm{II}}{}_{\mathrm{t-1}}\hat{\upsilon}^{\mathrm{I}\circ\mathrm{II}}_{\mathrm{II}}{}_{\mathrm{t-1}}\ldots \mathrm{S}^{\mathrm{I}\circ\mathrm{II}}{}_{\mathrm{t-i}}\left\{\mathrm{S}^{\mathrm{I}\circ\mathrm{II}}\right\}_{\mathrm{t-i}}\ldots{}^{\mathrm{i}}\hat{o}^{\mathrm{I}\circ\mathrm{II}}{}_{\mathrm{t-(\Delta t+1)}}\hat{\upsilon}^{\mathrm{I}\circ\mathrm{II}}{}_{\mathrm{t-(\Delta t+1)}} \in \Xi^{\mathrm{I}\circ\mathrm{II}}_{\mathrm{o}} \\ \mathbf{U}^{\mathrm{I}\otimes\mathrm{II}}_{\mathrm{t-i}} \in \mathfrak{V}^{\mathrm{I}\otimes\mathrm{II}} \end{gathered} \quad [5\text{-}64]$$

such that, for

$$\mathbf{W}^{\mathrm{I}\otimes\mathrm{II}} = \left| {}^{\mathrm{t-(\Delta t+1)}}_{\mathrm{I}\circ\mathrm{II}}\hat{o}_{\mathrm{i}} \right\rangle \left\langle {}^{\mathrm{i}}\hat{o}^{\mathrm{I}\circ\mathrm{II}}{}_{\mathrm{t-(\Delta t+1)}} \right| \quad [5\text{-}65]$$

$$\mathbf{Tr}^{\mathrm{I}\otimes\mathrm{II}}\left|\left[{}^{\mathrm{i}}o^{\mathrm{I}\circ\mathrm{II}}_{\mathrm{I}}{}_{\mathrm{t}}\right]^{\mathrm{I}\otimes\mathrm{II}}\right|\left[{}^{\mathrm{j}}o^{\mathrm{I}\circ\mathrm{II}}_{\mathrm{I}}{}_{\mathrm{t-1}}\right]^{\mathrm{I}\otimes\mathrm{II}}\left|\mathbf{U}^{\mathrm{I}\otimes\mathrm{II}}_{\mathrm{t-\Delta t}}\right|\ldots\left|\mathbf{U}^{\mathrm{I}\otimes\mathrm{II}}_{\mathrm{t-2}}\right|\mathbf{W}^{\mathrm{I}\otimes\mathrm{II}}\left|\hat{\mathbf{U}}^{\mathrm{I}\otimes\mathrm{II}}_{\mathrm{t-2}}\right|\ldots\left|\hat{\mathbf{U}}^{\mathrm{I}\otimes\mathrm{II}}_{\mathrm{t-\Delta t}}\right| = \delta^{\mathrm{i}}_{\mathrm{j}}. \quad [5\text{-}66]$$

Thus, in entanglement models

$$\mathbf{p}\left({}^{\mathrm{i}}\underset{\sim\mathrm{I}}{o}^{\mathrm{I}\circ\mathrm{II}}{}_{\mathbf{t}} \middle| \underset{\sim\mathrm{I}}{\hat{\upsilon}}^{\mathrm{I}\circ\mathrm{II}}{}_{\mathrm{t}}\,{}^{\mathrm{j}}\underset{\sim\mathrm{II}}{o}^{\mathrm{I}\circ\mathrm{II}}{}_{\mathrm{t-1}}\,\underset{\sim\mathrm{II}}{\hat{\upsilon}}^{\mathrm{I}\circ\mathrm{II}}{}_{\mathrm{t-1}}\ldots\underset{\sim}{\mathrm{S}}^{\mathrm{I}\circ\mathrm{II}}{}_{\mathrm{t-i}}\left\{\underset{\sim}{\mathrm{S}}^{\mathrm{I}\circ\mathrm{II}}\right\}_{\mathrm{t-i}}\ldots{}^{\mathrm{i}}\underset{\sim}{\hat{o}}^{\mathrm{I}\circ\mathrm{II}}{}_{\mathrm{t-(\Delta t+1)}}\,\underset{\sim}{\hat{\upsilon}}^{\mathrm{I}\circ\mathrm{II}}{}_{\mathrm{t-(\Delta t+1)}} \right) = \delta^{\mathrm{i}}_{\mathrm{j}}. \quad [5\text{-}67]$$

Here the observables $\hat{\upsilon}^{\mathrm{I}\circ\mathrm{II}}_{\mathrm{I}}{}_{\mathrm{t}}$ and $\hat{\upsilon}^{\mathrm{I}\circ\mathrm{II}}_{\mathrm{II}}{}_{\mathrm{t-1}}$ are members of $|\mathbf{O}^{\mathrm{I}\circ\mathrm{II}}|$ corresponding to component system maximal observables $\hat{\upsilon}_{\mathrm{I}}$ and $\hat{\upsilon}_{\mathrm{II}}$ in $|\hat{\mathbf{O}}^{\mathrm{I}}|$ and $|\hat{\mathbf{O}}^{\mathrm{II}}|$ respectively. These corresponding compound system observables are not maximal, hence their results are not decorated with overscript '^'. The '**i**' and '**j**' left superscripts index the results of observables $\hat{\upsilon}_{\mathrm{I}}$ and $\hat{\upsilon}_{\mathrm{II}}$. For simplicity, we assume these

$$\# \hat{\upsilon}_{I} = \# \hat{\upsilon}_{II} \quad [5\text{-}68]$$

so that [E-5-66] establishes a one-one correspondence between the results of these observables.

Here mathematical details about component system observables and the projection operators [ ]-corresponding to their results have been omitted.

Intuitively, for some initial result-observation events

$$^{i}\underset{\sim}{\hat{o}}^{I\circ II}{}_{t\text{-}(\Delta t+1)}\, \underset{\sim}{\hat{\upsilon}}^{I\circ II}{}_{t\text{-}(\Delta t+1)} \quad [5\text{-}69]$$

there exists a sequence of dynamic operators from $\mathfrak{V}^{I\otimes II}$ that carries the projection operator corresponding to this event

$$\left|\left[{}^{i}\hat{o}^{I\circ II}{}_{t\text{-}(\Delta t+1)}\right]\right| {}^{t\text{-}(\Delta t+1)}_{I\circ II}\hat{o}_{i}\rangle\langle {}^{i}\hat{o}^{I\circ II}{}_{t\text{-}(\Delta t+1)}\right| = \mathbf{W}^{I\otimes II} \quad [5\text{-}70]$$

into a projection operator

$$\left|\mathbf{U}^{I\otimes II}_{t\text{-}\Delta t}\right|\ldots\left|\mathbf{U}^{I\otimes II}_{t\text{-}2}\right|\mathbf{W}^{I\otimes II}\left|\hat{\mathbf{U}}^{I\otimes II}_{t\text{-}2}\right|\ldots\left|\hat{\mathbf{U}}^{I\otimes II}_{t\text{-}\Delta t}\right| \quad [5\text{-}71]$$

such that two subsequent projection operations

$$\left|\left[{}^{i}o^{I\circ II}_{I}{}_{t}\right]^{I\otimes II}\right|\left[{}^{j}o^{I\circ II}_{I}{}_{t\text{-}1}\right]^{I\otimes II} \quad [5\text{-}72]$$

corresponding to result-observation events

$$^{i}\underset{\sim I}{o}^{I\circ II}{}_{t}\, \underset{\sim I}{\hat{\upsilon}}^{I\circ II}{}_{t} \quad \text{and} \quad ^{j}\underset{\sim II}{o}^{I\circ II}{}_{t\text{-}1}\, \underset{\sim II}{\hat{\upsilon}}^{I\circ II}{}_{t\text{-}1} \quad [5\text{-}73]$$

yield a projection operator whose $\mathbf{Tr}^{I\otimes II}$-value is the delta function δ^{i}_{j}.

Still more intuitively, the conditional probability [E-5-67] may be understood to mean that knowing the result of observation event $\underset{\sim II}{\hat{\upsilon}}^{I\circ II}{}_{t\text{-}1}$ on a system $\mathbf{S}^{I\circ II}$ in **QM**-state $\left| {}^{t\text{-}(\Delta t+1)}_{I\circ II}\hat{o}_{i}\right\rangle$ allows one to predict with certainty the result of observation event $\underset{\sim I}{\hat{\upsilon}}^{I\circ II}{}_{t\text{-}1}$ on the system in the **QM**-state resulting from the first observation event.

That entangled models for **QM** exist is a mathematical fact demonstrated by the example of the **QM**-counterpart of Bell states (Nielson & Chuang 2000, pp. 25-26). The counter intuitive features of Bell states largely escape capture in the **QM** framework due to the absence of spatial concepts. Quantum mechanical counterparts of **QM** entangled states play a key role in theoretical discussions of quantum computing and physical realizations of them have been produced.

6. Constraints: C[QM𝔖]

Four constraints,

$$\mathbf{C}_{i} \subset \mathrm{POT}(\mathbf{M_p}[\mathbf{QM}\mathfrak{S}]) \quad [6\text{-}1]$$

i ∈ [0, 1, 2, 3] will be considered informally, so that

$$\mathbf{C}[\mathbf{QM}\mathfrak{S}] = \bigcap_i C_i \,. \qquad [6\text{-}2]$$

$\mathbf{C}_0$ requires that systems in $\mathbb{S}$ can be regarded as "real" physical systems in that they can not appear more than one place members of $\mathbf{C}_0$. $\mathbf{C}_1$ requires that the theoretical apparatus of **QM** assigned to systems be regarded an "intrinsic property" of ≈-equivalence classes of these systems in the sense ≈-equivalent systems are assigned the same theoretical apparatus wherever they appear in members of $\mathbf{C}_1$. It appears that $\mathbf{C}_0$ and $\mathbf{C}_1$ should be taken to apply to all intended applications of **QM**. That is, it should appear in the basic or "top" theory element in the specialization "tree" for **QM**.

$\mathbf{C}_2$ requires that whenever ○-concatenations of elementary systems equivalent to those appearing in members of $\mathfrak{M}_p \in \mathbf{C}_2$ appear as concatenates in another member of $\mathfrak{M}_p$, the Hilbert space $\mathbb{H}$ assigned to the concatenation is the tensor product of the $\mathbb{H}$'s assigned to the concatenates. Further, the [ ]-embedding works in such a way that

$$[o_{\mathrm{I}}]^{\mathrm{I}} \otimes [o_{\mathrm{II}}]^{\mathrm{II}} = [o_{\mathrm{I}} \circ o_{\mathrm{II}}]^{\mathrm{I}\otimes\mathrm{II}} \qquad [6\text{-}3]$$

where the details of the tensor product notation have not been explained here. This is somewhat analogous to the requirement in classical collision mechanics that mass be extensive with respect to concatenation (Balzer, Moulines & Sneed 1987, pp. 105-106).

$\mathbf{C}_3$ requires, in addition, that the [ ]-embedding into these tensor product spaces have certain symmetry properties with respect to certain ≈-equivalence classes of systems. Describing these symmetry properties requires some general concepts pertaining to permutations of sets and identifying certain ≈-equivalence classes of the set systems $\mathbb{S}$, analogous to Bosons and Fermions.

This leads to the definition of a general concept of [ ]-symmetry of which concepts analogous to those commonly encountered in quantum mechanics appear as special cases.

It appears that $\mathbf{C}_2$ and $\mathbf{C}_3$ are less general than $\mathbf{C}_0$ and $\mathbf{C}_1$ in that they might pertain only to certain kinds of systems.

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